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Globally hyperbolic flat space–times

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Abstract

We consider (flat) Cauchy-complete GH space–times, i.e., globally hyperbolic flat Lorentzian manifolds admitting some Cauchy hypersurface on which the ambient Lorentzian metric restricts as a complete Riemannian metric. We define a family of such space–times—model space–times—including four subfamilies: translation space–times, Misner space–times, unipotent space–times, and Cauchy-hyperbolic space–times (the last family—undoubtful the most interesting one—is a generalization of standard space–times defined by G. Mess). We prove that, up to finite coverings and (twisted) products by Euclidean linear spaces, any Cauchy-complete GH space–time can be isometrically embedded in a model space–time, or in a twisted product of a Cauchy-hyperbolic space–time by flat Euclidean torus. We obtain as a corollary the classification of maximal GH space–times admitting closed Cauchy hypersurfaces. We also establish the existence of CMC foliations on every model space–time.

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1. Introduction

Results concerning flat Lorentzian manifolds in the mathematical literature are mostly devoted to the case of *closed* manifolds (i.e. compact without boundary). And as a byproduct

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of this activity, the structure of closed flat manifolds has been elucidated in a quite satisfying way: see, for example, [3] for a quick survey on this problem.

But the non-compact case remains essentially open and non-considered, with the notable exception of the works related to Margulis space–times emerging from a question by Milnor concerning the (non)solvability of discrete groups acting on Minkowski space (see also [3]).

On the other hand, there is an important and natural notion in Lorentzian geometry: the *global hyperbolicity*. This notion is central in physics, in the area of classical General Relativity. And it is incompatible with compactness: a globally hyperbolic Lorentzian manifold is never compact. Because of this physical background, besides the notational convenience it provides, we call (flat) Lorentzian manifolds (flat) space–times.

A globally hyperbolic (abbreviation GH) space–time is a space–time M admitting a Cauchy hypersurface, i.e. a hypersurface S which:

- is spacelike (i.e., the Lorentzian metric restricts on S as a Riemannian metric),
- disconnects M,
- intersects every unextendible nonspacelike curve (i.e., for which every tangent vector has nonpositive norm) (see Section 2.4 for a more complete definition).

(M, S) is a *maximal* GH space–time (abbreviation MGH) if the only GH space–time containing M and for which S is still a Cauchy hypersurface, is M itself. A fundamental theorem by Choquet-Bruhat–Geroch [12] states than every GH pair (M, S) can be extended in a unique way to a maximal GH space–time N in which S is still a Cauchy hypersurface.

It is quite surprising that the classification of flat MGH space–times, which is the central topic here, has not been previously systematically undertaken. Such a classification has its physical interest, and even more, a mathematical one. It appears as a general extension of Bieberbach's theory to the Lorentzian context.

Actually, such a classification is not possible without some additional requirement: we will only consider here Cauchy-complete GH space–times, i.e., GH space–times admitting a Cauchy hypersurface on which the Lorentzian ambient metric restricts as a *complete* Riemannian metric. Among Cauchy-complete GH space–times, we distinguish the important subfamily of Cauchy-compact ones, for which the Cauchy hypersurface is closed. Cauchy-complete GH space–times which cannot be isometrically embedded in any bigger Cauchy-complete GH space–times, with no additional restriction on Cauchy surfaces, is *absolutely maximal* (abbreviation: AMGH space–times).

The most obvious examples of Cauchy-complete MGH space–times are simply quotients of $\mathbf{M}^n$ itself by discrete groups of spacelike translations. We call these examples *translation space–times*.

Next, there is another natural family that we call *Misner space–times* (see Section 3.2): there are quotients of the future in $\mathbf{M}^n$ of a spacelike $(n-2)$-subspace P by an abelian discrete group whose elements all admit as linear part a boost (or loxodromic element, maybe trivial) acting trivially in P, and translation part in P.

There is also the family of *unipotent space–times*, described in Section 3.3: let us briefly mention here that each of these space–times is the quotient of domain Ω in $\mathbf{M}^n$ delimited by one or two parallel degenerate hyperplanes by an abelian discrete group of unipotent isometries of $\mathbf{M}^n$.

Finally, and maybe the most interesting one, we have the family of what we call here *Cauchy-hyperbolic space–times*. In this family of examples, the linear part $L : \Gamma \to \mathrm{SO}_0(1, n-1)$ of the holonomy morphism will be injective, with image a discrete subgroup of $\mathrm{SO}(1, n-1)$ without torsion. Therefore, we consider $\Gamma \approx L(\Gamma)$ directly as a discrete subgroup of $\mathrm{SO}(1, n-1)$. We moreover assume Γ nonelementary, meaning that the orbits of its action on $\mathbb{H}^{n-1}$ are all infinite.

We first consider the case where Γ is a cocompact lattice of $\mathrm{SO}_0(1, n-1)$. Let Ω^+ be the connected component of $\{Q < 0\}$ geodesically complete in the future, and Ω^- the other connected component: Ω^- is geodesically complete in the past. The action of Γ on $\Omega^\pm$ is free and properly discontinuous: we denote by $M^\pm(\Gamma)$ the quotient manifold. Every level set $\{Q = -t^2\} \cap \Omega^\pm$ is Γ-invariant; it induces in $M^\pm(\Gamma)$ a hypersurface with induced metric of constant sectional curvature $-1/t^2$. Recall that $\{Q = -1\}$ is the usual representant of the hyperbolic space, therefore, the flat Lorentzian metric on $M^\pm(\Gamma)$ admits the warped product form $-\mathrm{d}t^2 + t^2 g_0$, where g_0 is the hyperbolic metric on $\Gamma \setminus \mathbb{H}^{n-1}$. We call these examples *radiant standard space–times*. Observe that $M^+(\Gamma)$ (resp. $M^-(\Gamma)$) is geodesically complete in the future (resp. in the past), and that there is a time reversing isometry between them.

In [16] or [2], it is shown that any representation of a Γ in $\mathrm{Isom}(\mathbf{M}^n)$ admitting as linear part an embedding onto a cocompact lattice of $\mathrm{SO}(1, n-1)$ preserves some future complete—and also a past complete-convex domain of $\mathbf{M}^n$, in such a manner that the quotients of these domains by $\rho(\Gamma)$ are Cauchy-compact AMGH space–times. These space–times are called by Mess and Andersson *standard space–times*.

When the linear part Γ is still a nonelementary discrete subgroup, but not cocompact in $\mathrm{SO}(1, n-1)$, the question is slightly more delicate. In Section 4, we extend the family of standard space–times to a more general one: the family of Cauchy-hyperbolic space–times (Definition 4.18). Briefly speaking, they are still quotients of semicomplete convex domains of $\mathbf{M}^n$ by discrete groups of isometries admitting as linear part a discrete subgroup. But, even in the radiant case, i.e. when $\rho(\Gamma)$ preserves a point in $\mathbf{M}^n$, let us say, the origin, the associated Cauchy-hyperbolic space–time is not always the quotient $\rho(\Gamma) \setminus \Omega^\pm$ as above, but most often some bigger space–time.

As a last comment on these examples, in Section 4.4.2 we prove that when Γ is a *convex cocompact Kleinian group*, then any discrete subgroup of $\mathrm{Isom}(\mathbf{M}^n)$ admitting Γ as linear part preserves a semicomplete domain of $\mathbf{M}^n$, i.e. is the holonomy group of some Cauchy-hyperbolic space–time. This claim is not as trivial as it may appear at first glance: in Proposition 4.22, we exhibit a (nonuniform) lattice Γ in $\mathrm{SO}_0(1, 2)$ such that, a representation $\rho : \Gamma \to \mathrm{Isom}(\mathbf{M}^n)$ for which $L \circ \rho$ is the identity morphism can preserve a semicomplete convex domain if and only if it is radiant, i.e. preserves a point in $\mathbf{M}^n$. This fuchsian group Γ is nothing but the group associated to the 3-punctured conformal sphere.

Convex cocompact Kleinian groups form an important family including cocompact Kleinian groups or Schottky groups, which are essentially the only examples appearing in the physical literature, except geometrically finite Kleinian groups (convex cocompact Kleinian groups can be defined as geometrically finite Kleinian groups without parabolic elements). The correct way to extend this result to geometrically finite Kleinian groups is an interesting question, even in the $2 + 1$-dimensional case.

Before stating the classification's theorems, let us indicate a natural way to produce GH space–times from other ones: given a flat GH space–time M, a flat Euclidean manifold N, and a representation $r : \pi_1(M) \to \mathrm{Isom}(N)$, the total space B of the suspension bundle over M with fiber N and monodromy r is naturally equipped with a flat Lorentzian metric which is still globally hyperbolic. The resulting space–time B from this construction, which is defined with more details in Section 5.2, is called the twisted product of M by N with monodromy r. The case where N is an Euclidean linear space and the representation r admits a global fixed point in N is particularly pleasant: we say then that B is a linear twisted product over M.

We now express the main theorem of this paper, essentially stating that the examples above provide the complete list of flat Cauchy-complete MGH space–times. In the following statement, a *tame embedding* is an isometric embedding inducing an isomorphism between fundamental groups.

Theorem 1.1. *Up to finite coverings and linear twisted products, every Cauchy-complete globally hyperbolic flat space–time can be tamely embedded in an absolutely maximal globally hyperbolic space–time which is a translation space–time, a Misner space–time, a unipotent space–time, or the twisted product of a Cauchy-hyperbolic flat GH space–time by an Euclidean torus.*

The Cauchy-compact case deserves its own statement.

Theorem 1.2. *Up to finite coverings, every maximal Cauchy-compact globally hyperbolic flat space–time is isometric to a translation space–time, a Misner space–time, or the twisted product of a standard space–time* (*Cauchy-compact Cauchy-hyperbolic GH space–time*) *by an Euclidean torus.*

The proofs of these theorems are written here in a quite intricate way, since it involves particular subcases to consider separately. Hence, we collect along the text all the elements of the proof, and then indicate in Section 11 the way to reconstruct from these intermediate results the complete proofs of the theorems.

Theorem 1.1 actually is not a full classification theorem: indeed, in a given MGH space–time we can embed many different MGH space–times-consider, for example, the Minkowski space itself! But, such a complete classification of Cauchy-complete GH space–times is not only unrealistic, it is furthermore useless when we adopt the point of view that the object of study here is the holonomy group Γ. In other words, our essential procedure is to associate to suitable discrete subgroups of $\mathrm{Isom}(\mathbf{M}^n)$ some invariant domains of $\mathbf{M}^n$ (regular convex domains) on which the dynamical behaviour of the group preserves some causality properties.

As a corollary of Theorem 1.2, we obtain that closed Cauchy hypersurfaces of globally hyperbolic flat space–times are homeomorphic to finite quotients of products of tori with hyperbolic manifolds. This generalizes [18] in any dimension, without the superfluous three-dimensional topological arguments used in [18].

There are other works related to the present work: in [19], Scannell classified Cauchy-compact maximal globally hyperbolic space–times with constant sectional curvature $+1$: the topological type of the Cauchy hypersurface S being fixed, there is a $1-1$ correspondence between MGH space–times and Riemannian flat conformal structures on S. Maximal globally hyperbolic Cauchy-compact space–times with constant curvature -1 of dimension $2+1$ are classified in [16]. We should mention that results in [18,16] are not stated in the terminology of GH space–times, but our presentation follows immediately from their works.

In [2], Andersson classified flat Cauchy-compact MGH space–times, but with the initial hypothesis that Cauchy hypersurfaces have hyperbolic type. He proves also that these space–times all admit foliations by constant mean curvature hypersurfaces (abbreviation, CMC foliation), which is unique in a given space–time. Thanks to Theorem 1.2, we can extend this result to the elementary case. It is done in Section 12.

The present paper includes also a generalization to any dimension of the characterization of "spacelike regions" for isometries of $\mathbf{M}^n$ made in [10] ("spacelike regions" of [10] are called here achronal domains).

2. Preliminaries

2.1. Space–times

By space–time, we mean here in general a (non-closed) oriented and chronologically oriented Lorentzian manifold. Our convention here is that Lorentzian metrics have signature $(-,+,\ldots,+)$. Tangent vectors are called *spacelike*, *timelike*, *lightlike* if their norms are respectively positive, negative, null. A curve in the space–time is nonspacelike if its tangent vectors have nonpositive norm. In the same spirit, a curve with timelike tangent vectors is called timelike, it has to be thought as a potential trajectory of a particle. Its proper time is defined by

$$\text{Proper-time}(c) = \int \sqrt{-\langle \partial_t c | \partial_t c\rangle}.$$

Since the space–time is chronologically oriented, every nonspacelike curve admits a canonical orientation towards its future.

A space–time is geodesically complete if every timelike geodesic admits geodesic parameterizations by $]-\infty,+\infty[$. It is *geodesically complete in the future* (resp., in the past) if every future oriented (resp. past oriented) timelike geodesic ray admits a geodesic parameterization by $]0,+\infty[$. Finally, a *geodesically semicomplete* space–time is a space–time which is either geodesically complete in the future or geodesically complete in the past.

2.2. Minkowski space

We denote here by $\mathbf{M}^n$ the Minkowski space. We stress out that we consider here $\mathbf{M}^n$ endowed with an orientation and a chronological orientation.

The Lorentzian quadratic form on $\mathbf{M}^n$ is denoted by $\langle\cdot|\cdot\rangle$; and the Minkowski norm is $|x|^2 = \langle x|x\rangle$ (the notation $|x|$ will be reserved to spacelike vectors). For any affine subspace E of $\mathbf{M}^n$, we denote by $E^\perp$ the orthogonal of E-this is a subspace of the underlying linear space. When E is a lightlike affine line, $E^\perp$ will also abusively denote the unique degenerate affine hyperplane containing E with direction orthogonal to E.

Let $L : \mathrm{Aff}(n, \mathbb{R}) \to GL(n, \mathbb{R})$ be the usual linear part morphism for the group of affine transformations of $\mathbf{M}^n$. The isometry group of the (unoriented) Minkowski space is the space of affine transformations with linear part in $\mathrm{O}(1, n-1)$, but since here the consider $\mathbf{M}^n$ as oriented and chronologically oriented, the term isometry will be reserved to affine transformations admitting linear parts in the identity component $\mathrm{SO}_0(1, n-1)$ (also called the orthochronous component). We denote by $\mathrm{Isom}(\mathbf{M}^n)$ the group of isometries of $\mathbf{M}^n$.

2.3. Flat space–times as geometric manifolds

Truly speaking, all space–times considered in this work are flat, i.e., locally modeled on the Minkowski space. In other words, in the language of geometric structures (see e.g. [15]), they are (G, X)-manifolds with $X = \mathbf{M}^n$ and $G = \mathrm{Isom}(M^n)$.

We will need only few facts on geometric structures: the existence of the developing map and the holonomy morphism (see Section 8).

2.4. Globally hyperbolic space–times

A space–time M is *globally hyperbolic* (abbreviation GH) if there is a proper time function $t : M \to \mathbb{R}$ such that every fiber $t^{-1}(t_0)$ is spacelike. Moreover, it is required that any nonspacelike curve in M can be extended to another one such that the restriction of t on this extended curve is a diffeomorphism onto the entire $t(M)$. In other words, every fiber of t is a *Cauchy hypersurface*, meaning precisely that this is a hypersurface intersecting every unextendible nonspacelike curve in M. There are other equivalent definitions of global hyperbolicity, see [4]. Observe that $t : M \to \mathbb{R}$ is necessarily a locally trivial, thus trivial fibration: M is diffeomorphic to $S \times \mathbb{R}$. Observe also that GH space–times admit chronological orientations.

Choquet-Bruhat–Geroch's work [12] implies that every GH manifold admits a unique maximal globally hyperbolic extension. Let us be more precise: let S be a Cauchy hypersurface in a globally hyperbolic space–time M. A S-embedding of M is an isometric embedding $f : M \to M'$ where M' is a space–time, such that $f(S)$ is still a Cauchy hypersurface in M'. Actually, this notion does not depend on the choice of the Cauchy hypersurface: if S, S' are Cauchy hypersurfaces in M, an isometric embedding $f : M \to M'$ is a S-embedding if and only if it is a S'-embedding. Therefore, we call such a map a *Cauchy embedding*. Now, a GH space–time M is said maximal (abbreviation MGH) if any Cauchy embedding of M into some GH space–time N is necessarily surjective. Choquet-Bruhat–Geroch's result can now be precisely stated: every GH space–time M admits a Cauchy embedding in a MGH space–time. Moreover, this maximal globally hyperbolic extension is unique up to isometries.

Remark 2.1. The maximal globally hyperbolic extension of a flat globally hyperbolic space–time is flat. Observe also that if the space–time is analytic, with analytic Lorentzian metric, then it is flat as soon as it contains a flat open set. In the definition above the time function may have low regularity (Lipschitz regularity is enough), but it can be proved that any GH space–time admits a smooth time function: we express here our gratitude to the referee for indicating the reference [7] where these regularity properties are discussed, with a valuable report on the history of these questions. Anyway, the regularity of Cauchy hypersurfaces is not relevant in the present work.

Remark 2.2. MGH space–times may admit nonsurjective isometric embeddings in bigger MGH space–times. Consider the following example: in $\mathbf{M}^n$, let Ω be a connected component of the negative cone $\{Q < 0\}$. Then, $S = \Omega \cap \{Q = -1\}$ is a Cauchy hypersurface for Ω. Observe that Ω is MGH, but nontrivially embedded in $\mathbf{M}^n$! The point is that S is not a Cauchy hypersurface in $\mathbf{M}^n$, therefore, the inclusion $\Omega \subset \mathbf{M}^n$ is not a Cauchy embedding.

2.5. Cauchy-complete, Cauchy-compact GH space–times

A GH space–time is *Cauchy-complete* if it admits a C^1 Cauchy hypersurface where the ambiant Lorentzian metric restricts as a complete Riemannian metric. This notion depends on the choice of the Cauchy hypersurface: for example, in the two-dimensional Minkowski space, i.e. the plane equipped with the metric $\mathrm{d}x\,\mathrm{d}y$, Cauchy hypersurfaces are graphs $y = f(x)$ of strictly increasing diffeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, and such a graph is complete if and only if the integrals of the square root of f' on $]-\infty, 0]$ and $[0, +\infty[$ are infinite.

Among Cauchy-complete space–times, there is a remarkable natural family: GH space–times with *closed* Cauchy hypersurfaces (we know that in general, if a space–time admits a closed Cauchy hypersurface, then all its Cauchy surfaces are closed-indeed, they are homeomorphic one to the other). We call such a space–time *Cauchy-compact* GH space–time (abbreviation CGH). It is worth to emphasize this family, since our results are much easier to state in this situation; keeping in mind our analogy with Bieberbach's theory on complete Euclidean manifolds, Cauchy-compact space–times correspond to closed Euclidean manifolds.

Remark 2.3. In the definition of Cauchy-completeness the Cauchy hypersurface is C^1. We don't know if in general Cauchy-complete space–times all admit smooth complete Cauchy hypersurfaces.

Remark 2.4. We say that a flat Cauchy-complete GH space–time M is absolutely maximal (abbreviation AMGH) if any isometric embedding into another Cauchy-complete flat GH space–time is surjective. Cauchy-compact MGH space–times are automatically absolutely maximal. Indeed, any spacelike hypersurface in a given CGH space–time is a Cauchy hypersurface.

3. Elementary Cauchy-complete AMGH space–times

3.1. Translation space–times

These are the easiest and more obvious examples, including $\mathbf{M}^n$ itself: they are the quotients of the entire $\mathbf{M}^n$ by a group of translations by spacelike translation vectors. There are orthogonal sums $S \oplus \mathbb{R}$, where S is a flat Euclidean cylinder $\mathbb{T}^k \times \mathbb{R}^{n-k-1}$ (where $\mathbb{T}^k$ is a flat torus of dimension k), and $\mathbb{R}$ is equipped with the metric $-\mathrm{d}t^2$.

3.2. Misner flat space–times

Consider the metric $2\,\mathrm{d}x\,\mathrm{d}y + \mathrm{d}z^2$ on $\mathbb{R}^n$, where z denotes an element of $\mathbb{R}^{n-2}$, and an isometry $\gamma_0(x, y, z) \mapsto (\mathrm{e}^{t_0}x, \mathrm{e}^{-t_0}y, z)$ $(t_0 \neq 0)$. The one-dimensional Misner space–time is the quotient of $\Omega = \{xy < 0, x > 0\}$ by the group generated by γ_0. Consider the pure loxodromic element γ_0 above as the time 1 of a 1-parameter subgroup γ_0^t, and let G_0 be the subgroup of $\mathrm{Isom}(\mathbf{M}^n)$ formed by elements with linear parts in γ_0^t and translations parts in the subspace $\{x = y = 0\}$. This an abelian Lie group isomorphic to $\mathbb{R}^{n-1}$, and the orbits of G_0 in Ω are all spacelike. If Γ is any discrete subgroup of G_0, the quotient $\Gamma \setminus \Omega$ is a flat globally hyperbolic space–time, that we call *Misner space–time*.

Observe that the linear part $L(\Gamma)$ is not necessarily discrete: it can be a dense subgroup of γ_0^t. Observe also that we could have selected $\Omega = \{xy < 0, x < 0\}$ too; but the two choices provide isometric space–times (up to a time reversing isometry), which are both geodesically semicomplete.

As a last comment on this example, we should add a particularly comfortable coordinate system: parameterize Ω by $x = \mathrm{e}^{\eta+\nu}$, $y = -\mathrm{e}^{\eta-\nu}$. The flat metric $2\,\mathrm{d}x\,\mathrm{d}y + \mathrm{d}z^2$ on Ω in these coordinates is

$$\mathrm{e}^{2\eta}(-2\,\mathrm{d}\eta^2 + 2\,\mathrm{d}\nu^2 + \mathrm{e}^{-2\eta}\,\mathrm{d}z^2).$$

The action of G_0 in this coordinate system is the action by translations on the z, ν coordinates.

3.3. Unipotent space–times

Consider once more a coordinate system (x, y, z) with x, y in $\mathbb{R}$ and z in $\mathbb{R}^{n-2}$, where the metric is given by $2\,\mathrm{d}x\,\mathrm{d}y + \mathrm{d}z^2$. We consider the unipotent part of the stabilizer of the lightlike hyperplane $\{y = 0\}$. This is the group $\mathcal{A}$ with elements of the form:

$$g_{u,v,\mu}(x, y, z) = (x + \mu - \langle z|v\rangle - \tfrac{1}{2}y|v|^2, y, z + u + yv),$$

where u and v are elements of $\mathbb{R}^{n-2}$, and μ a real number. The orbits of $\mathcal{A}$ are the degenerate hyperplanes $y = Cte$. Observe that $\mathcal{A}$ is not abelian, but a central extension of $\mathbb{R}$ by $\mathbb{R}^{2n-4}$: the operation law is:

$$g_{u,v,\mu} \circ g_{u',v',\mu'} = g_{u+u',v+v',\mu+\mu'-\langle u'|v\rangle}$$

Select now $n-2$ real numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-2}$, and an orthogonal basis $e_1, e_2, \ldots, e_{n-2}$ of the Euclidean space $\mathbb{R}^{n-2}$. Then, when $(t_1, \ldots, t_{n-2})$ describe the entire $\mathbb{R}^{n-2}$, the $g_{u,v,\mu}$ with $u = \sum t_i e_i$, $v = \sum t_i\lambda_i e_i$ and $\mu = -\sum t_i^2(\lambda_i/2)$ describe an abelian subgroup A of $\mathcal{A}$ isomorphic to $\mathbb{R}^{n-2}$.

The action is given by

$$\begin{aligned}(t_1, \ldots, t_{n-2}) \cdot \left(x, y, \sum z_i e_i\right)\\ = \left(x - \sum \frac{\lambda_i}{2}t_i^2 - \sum \lambda_i z_i t_i - \frac{y}{2}\sum \lambda_i^2 t_i^2, y, \sum(z_i + t_i(1 + y\lambda_i))e_i\right).\end{aligned}$$

The domain of $\mathbf{M}^n$ where the orbits of A are spacelike is $\Omega(A) = \{(x, y, z)/1 + y\lambda_i \neq 0\ \forall i\}$.

Define $y_i = -1/\lambda_i$ ($y_i = \infty$ if $\lambda_i = 0$), and $y_0 = -\infty \leq y_1 \leq \cdots y_{n-2} \leq y_{n-1} = +\infty$. Then, a connected component Ω of $\Omega(A)$ is the set of (x, y, z) with $y_j < y < y_{j+1}$ for some index j.

Consider such a connected component, and a discrete subgroup Γ of A. Observe that the boundary of Ω is the union of one or two lightlike hyperplanes $\{y = Cte\}$.

Let $f : I =]y_j, y_{j+1}[\to \mathbb{R}$ be any (strictly) increasing function. When y describes I, $(f(y), y, 0)$ describes a spacelike curve L in Ω. The saturation of L under the action of A is the set of (x, y, z) with:

- $z = \sum t_i(1 + \lambda_i y)e_i$,
- $x = f(y) - \sum(\lambda_i/2)t_i^2 - (y/2)\sum \lambda_i^2 t_i^2$.

Thus, it is the graph of the function $\Phi(y, z) = f(y) + (1/2)\sum z_i^2/(y_i - y)$ (with the conventions $z = \sum z_i e_i$, and $z_i^2/(y_i - y) = 0$ if $y_i = \infty$). Of course, S_f is a spacelike A-invariant hypersurface. More precisely, the Minkowski norm $2\,\mathrm{d}x\,\mathrm{d}y + \sum \mathrm{d}z_i^2$ induces on S_f the metric:

$$2f'(y)\,\mathrm{d}y^2 + \sum \left(\mathrm{d}z_i - \frac{z_i\,\mathrm{d}y}{y - y_i}\right)^2$$

with the convention $z_i\,\mathrm{d}y/(y - y_i) = 0$ if $y_i = 0$. If $\zeta_i = z_i/(y - y_i)$ (convention: $\zeta_i = z_i$ if $y_i = \infty$), the expression in the $(y, \zeta_1, \ldots, \zeta_{n-2})$ coordinates of the metric on S is (with the convention $(y - y_i)\,\mathrm{d}\zeta_i = \mathrm{d}\zeta_i$ if $y_i = \infty$):

$$2f'(y)\,\mathrm{d}y^2 + \sum (y - y_i)^2\,\mathrm{d}\zeta_i^2.$$

This is time now to specify the function f: let y_0 be any element of I, and $a : I \to]1, +\infty[$ any smooth function such that the integrals $\int_{y_j}^{y_0} a(y)\,\mathrm{d}y$ and $\int_{y_0}^{y_{j+1}} a(y)\,\mathrm{d}y$ are both infinite. Choose $f(y) = \int_{y_0}^{y} a^2(y)\,\mathrm{d}y$. We claim that for such a f, the metric on S_f is complete: if not, there is a smooth path $c : [0, +\infty[\to S_f$ with finite length and escaping to infinity. We write $c(t) = (y(t), \zeta_1(t), \ldots, \zeta_{n-2})$. Since the length is finite, the integrals $\int_0^{+\infty} \sqrt{2f'(y(t))}|y'(t)|\,\mathrm{d}t$ and $\int_0^{+\infty} \sqrt{\sum (y(t) - y_i)^2 \zeta_i'(t)^2}\,\mathrm{d}t$ are both finite. The finiteness of the first integral implies the finiteness of $\int_0^{+\infty} a(y(t))y'(t)\,\mathrm{d}t$. It follows that the y-coordinate remains in a compact subinterval of I. But, then, the integral $\int_0^{+\infty} \sqrt{\sum \zeta_i'(t)^2}\,\mathrm{d}t$ is bounded. It follows that $c(t)$ stay in a compact domain of S_f: contradiction.

Hence, S_f is a complete spacelike hypersurface.

We now prove that S_f is a Cauchy hypersurface for Ω. Clearly, the complement of S_f in $\mathbf{M}^n$ has two connected components, one containing $\{y = y_j\}$, the other containing $\{y = y_{j+1}\}$. Now, a nonspacelike geodesic in Ω is the intersection between Ω and a nonspacelike geodesic line d of $\mathbf{M}^n$. There are two cases to consider: if the direction of d is the isotropic direction Δ_0, then d must intersect S_f since S_f is a graph $x = \Phi(y, z)$. If not, d intersect the two lightlike hyperplanes $\{y = y_j\}$, $\{y = y_{j+1}\}$. Then, it must intersect S_f since S_f disconnects these hyperplanes.

Hence, any nonspacelike geodesic intersect S_f: this is a criterion proving that S_f is indeed a Cauchy hypersurface.

It follows that the quotient of Ω by Γ is globally hyperbolic and Cauchy-complete. We call such a space–time *unipotent space–time*. Unipotent space–times *never* have closed Cauchy hypersurfaces, since the $\mathcal{A}$-invariant coordinate y defines a submersion from the Cauchy hypersurface into $\mathbb{R}$. Observe also that unipotent space–times are not uniquely defined by their holonomy: indeed, the same Γ may be extended to a $A \approx \mathbb{R}^{n-2}$ in different ways. Moreover, even when A is fixed, different choices of connected components of $\Omega(A)$ lead to nonisometric unipotent space–times. Finally, unipotent space–times are not geodesically semicomplete, except those corresponding to connected components Ω with only one boundary hyperplane.

4. The non-elementary GH space–times: Cauchy-hyperbolic space–times

4.1. The Penrose boundary

In this section, we describe $\mathcal{J}$, the space of lightlike (or degenerate) affine hyperplanes of $\mathbf{M}^n$. We represent as usual $\mathbf{M}^n$ as $\mathbb{R}^n$ equipped with the metric $-\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n-1}^2$. We will also use the Euclidean metric $\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n-1}^2$; the norm of a vector v of $\mathbb{R}^n$ for this metric will be denoted $N(v)$.

Denote by $\mathcal{S}$ the space of linear lightlike hyperplanes. It admits a natural $1-1$-parameterization by vectors v of $\mathbb{R}^n$ for which $N(v) = 1$, $|v| = 0$, and v oriented towards the future, i.e., $x_0 > 0$. We denote by $\mathcal{S}^+$ the space of such vectors. A vector in $\mathcal{S}^+$ is completely characterized by its $x_1, \ldots, x_{n-1}$ coordinates which satisfy $\sum x_i^2 = 1/2$ (observe that necessarily $x_0 = 1/2$)). Hence, $\mathcal{S}^+$ is naturally identified with the $(n-2)$-sphere. The ambient Lorentzian metric on $\mathbf{M}^n$ induces on $\{N = 1, |\cdot| = 0\}$ a non-degenerate Riemannian metric, and an easy computation proves that this metric is nothing but the usual round metric on the $(n-2)$-sphere. In the same vein, $\mathcal{S}$ can also parameterized by $\mathcal{S}^-$, the space of *past oriented* lightlike vectors with N-norm 1.

The group $\mathrm{SO}_0(1, n-1)$ acts naturally on $\mathcal{S}$ but it does not preserve the "round metric" we have just defined-mainly because it does not preserves $N = 1$. Actually, for g in $\mathrm{SO}_0(1, n-1)$, and identifying $\mathcal{S}^\pm$ with the space of v's as above, the action of g on $\mathcal{S}$ maps v on $[g] \cdot v = g(v)/N(g(v))$. This action preserves the *conformal class* of the metric; in particular, it preserves the angles. As a matter of fact, $\mathcal{S}^\pm$ equipped with this conformal class is the usual conformal sphere, and elements of $\mathrm{SO}_0(1, n-1)$ act on it as Möbius transformations.

The space $\mathcal{J}$ admits a natural fibration over each $\mathcal{S}^\pm$: this map is just the map associating to the affine hyperplane its direction. We denote it by $\delta^\pm : \mathcal{J} \to \mathcal{S}^\pm$. Now, for any element P of $\mathcal{J}$ with $v = \delta^\pm(P)$, the scalar product $\langle v|p\rangle$ does not depend on the point p of P: we denote it by $\nu^\pm(P)$. Then, the product map $\delta^\pm \times \nu^\pm : \mathcal{J} \to \mathcal{S}^\pm \times \mathbb{R}$ is one-to-one.

Remark 4.1. The ±-ambiguity means that we actually have two natural identifications of $\mathcal{J}$ with the product of the conformal sphere by the real line. We will indicate which of these identifications maps $\delta^{\pm} \times \nu^{\pm}$ is considered by denoting the space of lightlike hyperplanes by $\mathcal{J}^{+}$ or $\mathcal{J}^{-}$. Alternatively, we can adopt the point of view that $\mathcal{J}^{+}$ (resp. $\mathcal{J}^{-}$) is the space of future complete (resp. past complete) half spaces bounded by lightlike hyperplanes.

Finally, Isom$(\mathbf{M}^n)$ acts naturally on $\mathcal{J}$. When we use the identification $\mathcal{J} \approx \mathcal{J}^{+} \approx \mathcal{S} \times \mathbb{R}$, the action of an isometry $g(x) = L(g)x + \tau$ is expressed by:

$$g \cdot (v, s) = ([L(g)] \cdot v, a(g, v)s + b(g, v)),$$

where

$$[L(g)] \cdot v = \frac{L(g)v}{N(L(g)v)}, \qquad a(g, v) = \frac{1}{N(L(g)(v))}, \qquad b(g, v) = \frac{\langle \tau | L(g)v \rangle}{N(L(g)v)}$$

Observe that for the other identification $\mathcal{J} \approx \mathcal{J}^{-}$, the action would be expressed in a similar form, with the same $a(g, v)$, but with an opposite $b(g, v)$. Observe also that the factor $1/N(L(g)v)$ is common to the v and s components. Hence, we have the following proposition:

Proposition 4.2. *Equip $\mathcal{J}^{+} \approx \mathcal{S}^{+} \times \mathbb{R}$ with the product metric of the round metric on the sphere $\mathcal{S}^{+}$ and any Euclidean metric on $\mathbb{R}$. Let g be a linear isometry of $\mathbf{M}^n$, i.e., admitting a trivial translation part τ. Then, the domain in $\mathcal{J}^{+}$ where the action of g is expanding is precisely the preimage under δ^{+} of the domain of $\mathcal{S}^{+}$ where the action of $L(g)$ is expanding.*

Remark 4.3. The title of this section is justified by the fact that $\mathcal{J}^{\pm}$ are nothing but the regular parts of the Penrose boundary of Minkowski space as usually defined in general relativity (and usually with the same notation, see e.g. [13]).

4.2. Regular convex domains

Let Λ be any subset of $\mathcal{J}$. We will always assume that Λ contains at least one point. For any element P of Λ, let P^{+} be the future of P, and P^{-} the past of P. These are half-spaces, admitting both P as boundaries. If $v = \delta^{+}(P)$ and $s = \nu^{+}(P)$, P^{+} (resp. P^{-}) is the domain of points p in $\mathbf{M}^n$ for which $\langle v | p \rangle - s$ is negative (resp. positive).

Definition 4.4. The future complete (resp. past complete) convex set defined by Λ is the intersection:

$$\Omega^{+}(\Lambda) = \bigcap_{P \in \Lambda} P^{+} \left(\text{resp. } \Omega^{-}(\Lambda) = \bigcap_{P \in \Lambda} P^{-} \right).$$

If Λ contains at least two elements, then $\Omega^{\pm}(\Lambda)$, if nonempty, is regular.

We first collect some straightforward observations:

- $\Omega^{\pm}(\Lambda)$ is an convex set.
- $\Omega^{+}(\Lambda)$ is (geodesically) complete in the future, and $\Omega^{-}(\Lambda)$ is complete in the past. They are disjoint one from the other.

Definition 4.5. If $\Omega^{+}(\Lambda)$ (resp. $\Omega^{-}(\Lambda)$) is nonempty and open, then it is called a future (resp. past) regular convex domain. The set Λ is then said future regular (resp. past regular).

This definition of regular convex domains is completely equivalent to the definition given by Bonsante (see [8], Definition 4.1), but it is essential for our purposes to relate it with closed subsets of $\mathcal{J}$.

Lemma 4.6. *If $\overline{\Lambda}$ is the closure in $\mathcal{J}$ of Λ, then $\Omega^{\pm}(\overline{\Lambda})$ contains the interior of $\Omega^{\pm}(\Lambda)$.*

Proof. Let x be a point in the interior of $\Omega^{+}(\Lambda)$. Assume that x does not belong to $\Omega^{+}(\overline{\Lambda})$. Then, for some element (v, s) of $\Omega^{+}(\overline{\Lambda})$, we have $\langle x|v\rangle \geq s$. On the other hand, for any sequence (v_n, s_n) in Λ converging to (v, s), we have $\langle x|v_n\rangle < s_n$. At the limit, we obtain: $\langle x|v\rangle = s$. But, since x is in the interior of $\Omega^{+}(\Lambda)$, there is a point y in the interior of $\Omega^{+}(\Lambda)$ near x for which $\langle y|v\rangle > s$. Apply once more the argument above to y: we obtain a contradiction. □

Corollary 4.7. *If Λ is future regular, then its closure $\overline{\Lambda}$ is future regular too, and $\Omega^{+}(\Lambda) = \Omega^{+}(\overline{\Lambda})$.*

Proof. The inclusion $\Omega^{+}(\overline{\Lambda}) \subset \Omega^{+}(\Lambda)$ is obvious. Hence, the corollary follows immediately from Lemma 4.6. □

Lemma 4.8. *If Λ is closed, then $\Omega^{+}(\Lambda)$ is open (possibly empty).*

Proof. Assume the existence of some element x belonging to $\Omega^{+}(\Lambda)$, but not to its interior. Then, there is a sequence of elements x_n in $\mathbf{M}^n$ converging to x, but not belonging to $\Omega^{+}(\Lambda)$. It means that for every n, there is an elements (v_n, s_n) in Λ for which $\langle x_n|v_n\rangle \geq s_n$. By compactness of $\mathcal{S}$, we can assume that the sequence v_n converges to some v. Since x belongs to $\Omega^{+}(\Lambda)$, we have: $\langle x|v_n\rangle < s_n$. It follows that s_n converges to $\langle x|v\rangle$. But since Λ is closed, it must contain the limit point $(v, \langle x|v\rangle)$. This is a contradiction since x is assumed to belong to $\Omega^{+}(\Lambda)$. □

Lemma 4.9. *Let Λ be a subset of $\mathcal{J}$ with future regular closure. Then, $\Omega^{+}(\overline{\Lambda})$ is the interior of $\Omega^{+}(\Lambda)$.*

Proof. According to Lemma 4.6, $\Omega^{+}(\overline{\Lambda})$ contains the interior of $\Omega^{+}(\Lambda)$. But it is obviously contained in $\Omega^{+}(\Lambda)$, and, according to Lemma 4.8, it is open. The lemma follows. □

Thanks to Corollary 4.7 and Lemma 4.8, we can define regular convex domains as *nonempty* convex sets associated to *closed* sets Λ in $\mathcal{J}$ not reduced to one point (observe that the similar statements for past regular domains are evidently valid).

Proposition 4.10. *A closed subset of $\mathcal{J}$ is future regular (resp. past regular) if and only if there is some real number C such that, for every (v, s) in Λ, the second component s is less than C (resp. bigger than C).*

Proof. Assume that Λ is contained in a domain $\{s > C\}$. Then, $\{s = C\}$ is a family of lightlike hyperplanes tangent to the future cone of some point x satisfying $\langle x|v\rangle = C$ for every future oriented lightlike v satisfying $N(v) = 1$. In particular, $\langle x|v\rangle < s$ for every (v, s) in Λ:x belongs to $\Omega^+(\Lambda)$ which therefore is nonempty.

Inversely, if x belongs to $\Omega^+(\Lambda)$, then, for every (v, s) in Λ, we have $\langle x|v\rangle < s$. Since $\mathcal{S}$ is compact, $\langle x|v\rangle$ admits a lower bound independent from v. The proposition follows (the past regular case is completely similar). □

Corollary 4.11. *Λ is future regular and past regular if and only if it is compact.*

In the following, we state many results proved in [8] about the *cosmological time of* $\Omega^\pm(\Lambda)$, without any attempt to provide proofs. We just suggest to the reader to keep in mind the similar situation in Euclidean space when is considered the distance function to some convex set.

For any point p in $\Omega^+(\Lambda)$, consider every future oriented timelike curve starting from a point in $\partial\Omega^+(\Lambda)$ and ending at p. The proper times of all these curves are uniformly bounded: let $T(p)$ be the supremum value of these proper times. There is a unique $\pi(p)$ in $\partial\Omega^+(\Lambda)$ for which $T(p)$ is equal to $-\sqrt{|p-\pi(p)|^2}$; in other words, the straight segment from $\pi(p)$ to p is the unique timelike curve joining p and $\partial\Omega^+(\Lambda)$ with proper time realizing $T(p)$. The function $T : \Omega^+(\Lambda) \to \mathbb{R}^+_*$ is a C^1-convex function (Proposition 4.3 in [8]). The level sets $S_t = T^{-1}(t)$ are convex spacelike hypersurfaces (Corollary 4.5 in [8]); actually, the direction of the tangent space to S_t at some point p is the orthogonal of $p - \pi(p)$. Hence, $n(p) = (p - \pi(p))/\sqrt{-\langle p-\pi(p)|p-n(p)\rangle}$ is the normal vector to S_t at p pointing in the future. For this reason, we call $n : S_t \to \mathbb{H}^{n-1}$ the *Gauss map*. Moreover, Corollary 4.5 expresses much more: $S(p + \epsilon n(p)) = S(p) + \epsilon$. Thus, from one level set, let us say, S_1, and the Gauss map n on it, we can reconstruct all other level sets.

Proposition 4.12. *The Gauss map $n : S_t \to \mathbb{H}^{n-1}$, where $\mathbb{H}^{n-1}$ is equipped with its usual hyperbolic metric is $1/t$-Lipschitz.*

Proof. Select any $0 < \epsilon < t$. When p describes S_t, then $q(p) = p - \epsilon(p)n(p)$ describe $S_{t-\epsilon}$, and $n(q(p)) = n(p)$. Since T is C^1 and has spacelike fibers, S_t admits an induced Riemannian metric, and for any p, p' in S_t, the distance between them is the infimum of $\sum_{i=0}^{N-1} |p_{i+1} - p_i|$ when $p_0, \ldots, p_N$ describe all the finite sequences in S_t with $p_0 = p$, $p_N = p'$. For such a sequence, we have:

$$|p_{i+1} - p_i|^2 = |q(p_{i+1}) - q(p_i)|^2 + \epsilon^2|n(q(p_{i+1})) - n(q(p_i))|^2 + 2\epsilon\langle q(p_{i+1}) - q(p_i)|n(q(p_{i+1})) - n(q(p_i))\rangle.$$

The convexity of $S_{t-\epsilon}$ implies the positivity of $\langle q(p_{i+1}) - q(p_i)|n(q(p_{i+1})) - n(q(p_i))\rangle$. Hence:

$$|n(p_{i+1}) - n(p_i)|^2 = |n(q(p_{i+1})) - n(q(p_i))|^2 \leq \frac{1}{\epsilon^2}|p_{i+1} - p_i|^2.$$

Since ϵ can be selected arbitrarily near t, the proposition follows. □

Corollary 4.13. *Every S_t is complete.*

Proof. Since S_t is Riemannian, completeness notions are all equivalent. Let $s \mapsto p(s)$ be an incomplete geodesic in S_t defined on $[0, s_\infty[$, parameterized by unit length. According to Proposition 4.12, $s \mapsto n(c(s))$ must converge in $\mathbb{H}^{n-1}$ to some limit point n_∞. We select a coordinate system on $\mathbf{M}^n$ so that this limit vector is $(1, 0, \ldots, 0)$ and such that the Minkowski norm is $-\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n-1}^2$. For some small α, every tangent vector $\partial_t c(t)$ with $s - \alpha < s < s_\infty$ has x_0-component less than, let us say, 1/2. It follows that the orthogonal projection on $P_0 = \{x_0 = 0\}$ of the geodesic c has finite length for the usual Euclidean metric of P_0. Hence, $c(s)$ has a limit point c_∞ in P_0. We claim that the vertical line above c_∞, like any timelike line in $\mathbf{M}^n$, intersects $\partial\Omega^+(\Lambda)$: indeed, it must enter in the future cone of every point in $\Omega^+(\Lambda)$. Since $\Omega^+(\Lambda)$ is geodesically complete in the future, the vertical line must thus intersect $\Omega^+(\Lambda)$. On the other hand, it cannot be entirely contained in $\Omega^+(\Lambda)$ since it intersects every degenerate hyperplanes, and thus, every element of Λ.

From the geodesic completeness in the future of $\Omega^+(\Lambda)$, it follows now that the vertical line intersects S_t at one and only one point p_∞. The initial geodesic can then be completed on $[0, s_\infty]$ by $c(s_\infty) = p_\infty$. □

Finally, as it is proved in [8], Lemma 4.9, every S_t is a Cauchy hypersurface for $\Omega(\Lambda)$. Thus, we have the following proposition.

Proposition 4.14. *Future complete regular convex domains are globally hyperbolic Cauchy-complete, admitting as Cauchy hypersurfaces the level sets of the cosmological time function.*

Remark 4.15. We did not systematically state all the similar results for past complete regular convex domains, but of course they are true.

We also could have written this section in a slightly different way, by defining future (resp. past) regular convex domains as defined by closed subsets of $\mathcal{J}^+$ (resp. $\mathcal{J}^-$) (see Remark 4.1), but it is useful for the next section to stress out that these closed subsets arise from closed subsets in the same space $\mathcal{J}$.

4.3. Groups preserving regular convex domains and Cauchy-hyperbolic space–times

In the first part of this section, Γ is a discrete subgroup of $\mathrm{Isom}(M^n)$.

Proposition 4.16. *If Γ preserves a future complete regular convex domain Ω, then, the action of $\rho(\Gamma)$ on Ω is properly discontinuous. If moreover Γ is torsionfree, then this action is free, and the quotient $\Gamma \setminus \Omega$ is a Cauchy-complete semicomplete GH space–time.*

Proof. According to the preceding section, Ω is globally hyperbolic, with a regular cosmological time function $T : \Omega \to]0, +\infty[$. This function is Γ-invariant. The action of Γ on every level set S_t is isometric for the induced Riemannian metric. It follows that the action on Ω is properly discontinuous. Observe that an element of Γ is trivial as soon as its action on a level set S_t is trivial. Indeed, for any x in such a S_t, γ is an isometry of $\mathbf{M}^n$ which is trivial on the spacelike hyperplane T_xS_t: since it preserves chronological orientation, it follows that γ is trivial.

Assume now that γ admits a fixed point x in Ω. Then, since its action on S_t with $t = T(x)$ is isometric, and since Γ is discrete, it follows that γ has finite order when restricted to S_t. According to the above, γ itself has then finite order. Hence, if Γ is torsionfree, the action is free. The quotient space $\Gamma \setminus \Omega$ is then a space–time, admitting the cosmological time function induced by T expressing it as a GH space–time. The level sets of this time function are Cauchy hypersurfaces which are quotients $\Gamma \setminus S_t$: the induced Riemannian metric is complete. □

From now, Γ denotes a discrete subgroup of $SO_0(1, n-1)$ without global fixed point on $\bar{\mathbb{H}}^{n-1}$. Let $\rho : \Gamma \to \mathrm{Isom}(\mathbf{M}^n)$ be a morphism such that $L \circ \rho$ is the identity map. We say that ρ is *future admissible* (resp. past admissible) if $\rho(\Gamma)$ preserves a future complete (resp. past complete) regular convex domain.

According to Proposition 4.16, admissible representations produce Cauchy-complete GH space–times. We want to associate to every such representation a *maximal* GH space–time; it will be provided by the following proposition. Before stating the proposition, it is maybe convenient for some readers to recall that a fixed point of a diffeomorphism is said *repulsive* if the derivative of the diffeomorphism at this fixed point is expanding, i.e., the inverse of the derivative has norm less than 1.

Proposition 4.17. *For any admissible representation ρ, the closure of the set of repulsive fixed points in $\mathcal{J}$ of loxodromic elements of $\rho(\Gamma)$ is a $\rho(\Gamma)$-invariant subset $\Lambda(\rho)$ contained in any closed $\rho(\Gamma)$-invariant subset of $\mathcal{J}$.*

Proof. Consider an admissible representation ρ, preserving a closed subset Λ. The fibration $\delta : \mathcal{J} \to \mathcal{S}$ is Γ-equivariant, where the Γ-action on $\mathcal{S}$ is its usual conformal action on the sphere. The dynamic of (nonelementary) discrete subgroups of $SO_0(1, n-1)$ on the conformal sphere is well-known: there is a closed subset $\hat{\Lambda}$, the limit set, contained in every Γ-invariant closed subset—in particular, in the closure of $\delta(\Lambda)$. It is also well-known that $\hat{\Lambda}$ is the closure of repulsive fixed points of loxodromic elements of Γ. Let γ such a loxodromic element of Γ, and v_0 the element of $\mathcal{S}$ corresponding to the repulsive fixed point of γ. We consider here v_0 as a future oriented lightlike vector in $\mathbf{M}^n$ with N-norm 1, then, v_0 is a γ eigenvector, with eigenvalue $0 < \lambda < 1$. The bassin of repulsion of x_0 for $\rho(\gamma)$ contains some open subset of the form $\delta^{-1}(U)$, where U is some open neighborhood of v_0 in $\mathcal{S}$. Then, U contains some element of $\delta(\Lambda)$, hence, $\delta^{-1}(U)$ contains some element x of Λ. Then, negative iterates of x under $\rho(\gamma)$ converge towards x_0.

It follows as required that every repulsive fixed point of loxodromic elements of $\rho(\Gamma)$ belongs to Λ. □

Definition 4.18. A Cauchy-hyperbolic space–time is the quotient of $\Omega^{\pm}(\Lambda(\rho))$ by $\rho(\Gamma)$ for any admissible representation $\rho : \Gamma \to \mathrm{Isom}(\mathbf{M}^n)$.

Remark 4.19. It follows from Proposition 4.17 and Section 11.3 that any Cauchy-complete GH space–time with holonomy group $\rho(\Gamma)$ can be isometrically embedded in the associated Cauchy-hyperbolic space–time.

4.4. Admissible and non-admissible representations

Let Γ be a nonelementary discrete subgroup of $SO_0(1, n-1)$. We consider the space of representations $\rho : \Gamma \to \mathrm{Isom}(\mathbf{M}^n)$ admitting the identity map as linear part. For such a ρ, the translation part $\tau(\gamma)$ in the expression $\rho(\gamma)x = \gamma x + \tau(\gamma)$ defines a map $\tau : \Gamma \to \mathbb{R}^{1,n-1}$

which satisfies $\tau(\gamma\gamma') = \tau(\gamma) + \gamma\tau(\gamma')$, i.e. which is a 1-cocycle for the Γ-module $\mathbb{R}^{1,n-1}$. We denote by $Z^1(\Gamma, \mathbb{R}^{1,n-1})$ the space of cocycles.

Two such cocycles defines representations conjugate by a translation of $\mathbf{M}^n$ if and only if they differ by a coboundary, i.e., a cocycle of the form $\tau(\gamma) = \gamma v - v$ for some fixed v. We denote by $B^1(\Gamma, \mathbb{R}^{1,n-1})$ the space of coboundaries.

In other words, the space of representations ρ as above up to conjugacy by translations is parameterized by the quotient of $Z^1(\Gamma, \mathbb{R}^{1,n-1})$ by $B^1(\Gamma, \mathbb{R}^{1,n-1})$, i.e., the twisted cohomology space $H^1(\Gamma, \mathbb{R}^{1,n-1})$, the Γ-module structure on $\mathbb{R}^{1,n-1}$ being given by the fixed linear part Γ. Observe that $H^1(\Gamma, \mathbb{R}^{1,n-1})$ is naturally equipped with a structure of linear space.

Here, we want to describe the set of *future admissible* representations, i.e. the domain $\mathcal{T}^+$ in $H^1(\Gamma, \mathbb{R}^{1,n-1})$ corresponding to representations preserving a future complete regular convex domain. According to Proposition 4.17, this is exactly the set of cocycles corresponding to representations for which the future complete convex set defined by repulsive fixed points of loxodromic elements is not empty.

Similarly, we can define the domain $\mathcal{T}^-$ corresponding to representations preserving some past complete regular domains, but observe that the conjugacy by $-id$ induces a transformation on $H^1(\Gamma, \mathbb{R}^{1,n-1})$, which is nothing but the antipodal map, and exchanges $\mathcal{T}^+$, $\mathcal{T}^-$. In other words, we have $\mathcal{T}^- = -\mathcal{T}^+$, hence, we restrict our study to future complete regular domains.

The conjugacy by some positive homothety λid, $\lambda > 0$ preserves the linear part too and induces on $H^1(\Gamma, \mathbb{R}^{1,n-1})$ a positive homothety. It clearly preserves $\mathcal{T}^+$ which is thus a cone.

Lemma 4.20. *$\mathcal{T}^+$ is a convex cone.*

Proof. Let $[\tau_1]$, $[\tau_2]$ be two elements of $\mathcal{T}^+$. Let us denote Λ_i the closure in $\mathcal{J}$ of the set of repulsive fixed points of loxodromic elements of Γ for the representation ρ_i associated to τ_i. Then, $\Omega^+(\Lambda_i)$ for $i = 1, 2$ are nonempty future complete regular domains, and thus, have nonempty intersection. Let p be an element of $\Omega^+(\Lambda_1) \cap \Omega^+(\Lambda_2)$.

For any loxodromic element γ of Γ, we denote by $(v(\gamma), s_i(\gamma))$ its unique repulsive fixed point in $\mathcal{J} \approx \mathcal{S} \times \mathbb{R}$ for the representation ρ_i. Observe that $v(\gamma)$ is indeed the same for $i = 1, 2$, since it is the future oriented γ-eigenvector with N-norm 1 associated to the eigenvalue λ with absolute value less than 1. We have (see Section 4.1):

$$(v(\gamma), s_i(\gamma)) = \rho_i(\gamma)(v(\gamma), s_i(\gamma)) = (v(\gamma), \lambda^{-1}s_i(\gamma) + \lambda^{-1}\langle\tau_i|\lambda v(\gamma)\rangle).$$

Hence:

$$s_i(\gamma) = -\frac{\langle\tau_i|v(\gamma)\rangle}{\lambda^{-1} - 1}.$$

Now, since p belongs to $\Omega^+(\Lambda_1) \cap \Omega^+(\Lambda_2)$, for any loxodromic γ and any i, we have $\langle v(\gamma)|p\rangle - s_i(\gamma) < 0$, i.e.:

$$(\lambda^{-1} - 1)\langle v(\gamma)|p\rangle + \langle\tau_i|v(\gamma)\rangle < 0.$$

Clearly, this last expression is still valid if the term τ_i is replaced by any $\alpha\tau_1 + (1-\alpha)\tau_2$ with $0 \leq \alpha \leq 1$. It follows that representations associated to $[\alpha\tau_1 + (1-\alpha)\tau_2]$ admits as set of repulsive fixed points in $\mathcal{J}$ a subset for which the associated future complete convex contains p, i.e., is not empty. This means precisely that these representations all belong to $\mathcal{T}^+$. □

Remark 4.21. The intersection $\mathcal{T}^+ \cap \mathcal{T}^-$ is a linear subspace since it is a convex cone stable by antipody. Observe that this intersection correspond to representations preserving a *compact* subset of $\mathcal{J}$ (cf. Corollary 4.11).

4.4.1. Non-admissible cocycles

Proposition 4.22. *There is a* (*nonuniform*) *lattice* Γ *of* $\mathrm{SO}_0(1,2)$ *for which* $\mathcal{T}^+$ *is reduced to* $\{0\}$.

Proof. Consider the 3-punctured sphere, i.e. the Riemann surface of genus 0 with three cusps (there is only one such Riemann surface). It is the quotient of the Poincaré disk by a fuchsian group Γ, which is isomorphic to the free group of rank 2, generated by three parabolic elements a, b and c of $\mathrm{SO}_0(1,2)$ satisfying the relation $abc = id$. There is another description of Γ: consider an ideal triangle in the Poincaré disk, and the group generated by reflexions around edges of this triangle. This group contains an index 2 subgroup, the orientation preserving elements, which is nothing but the group Γ. It is clear from this last description that there is an elliptic element R of $\mathrm{SO}_0(1,2)$ of order 3-the rotation permuting the ideal vertices of the initial ideal triangle-such that the conjugacy by R cyclically permutes a, b and c. Denote by α, β and κ the unique isotropic vectors fixed respectively by a, b and c (they are the vertices of the initial ideal triangle). Of course, $\kappa = R(\beta) = R^2(\alpha)$.

Consider now $H^1(\Gamma, \mathbb{R}^{1,2})$. For any cocycle τ, every $\tau(\gamma)$ can be computed from $\tau(a)$ and $\tau(b)$, since a and b generate Γ. Hence, cocycles form a six-dimensional linear space naturally identified with $\mathbb{R}^{1,2} \times \mathbb{R}^{1,2}$. Coboundaries are the image of the map from $\mathbb{R}^{1,2}$ into $\mathbb{R}^{1,2} \times \mathbb{R}^{1,2}$ which associates to x the pair $(ax - x, bx - x)$. This map is injective, since a, b have no (nontrivial) common fixed points, hence, the image is three-dimensional: $H^1(\Gamma, \mathbb{R}^{1,2})$ has dimension 3.

It will be useful later to represent elements of $Z^1(\Gamma, \mathbb{R}^{1,2})$ by triples $(\tau(a), \tau(b), \tau(c))$ satisfying the cocycle relation $\tau(a) + a\tau(b) + ab\tau(c) = 0$.

Now, if τ belongs to $\mathcal{T}^+$, the Minkowski isometries associated by this cocycle to a, b and c all preserve a regular convex domain, and thus, a spacelike complete hypersurface. According to Section 7, these isometries are not transverse, i.e., the translation vectors $\tau(a)$, $\tau(b)$ and $\tau(c)$ are respectively orthogonal to α, β and κ. Denote by E the space of cocycles satisfying these orthogonality conditions: we have just proved that E contains $\mathcal{T}^+$. Observe that E is defined by linear conditions on $H^1(\Gamma, \mathbb{R}^{1,2})$; it is therefore a linear subspace. Moreover, since $\tau(a)$, $\tau(b)$ can be selected non-orthogonal to α, β, the codimension of E is at least 2. At first glance, one could think that the third condition "$\tau(c)$ is orthogonal to κ" immediately implies that E has codimension 3, i.e., is trivial. But this is not clear since it could be true that by some extraordinary miracle, $\tau(c)$ is orthogonal to κ as soon as $\tau(a)$, $\tau(b)$ are themselves orthogonal to α, β. In this case, E would be a line.

To prove that this miracle does not occur, we consider the rotation R above: the conjugacy by R induces naturally an action on $Z^1(\Gamma, \mathbb{R}^{1,2})$: if τ is a cocycle, $R(\tau)(\gamma) = R\tau(R^{-1}\gamma R)$. When we express elements of $Z^1(\Gamma, \mathbb{R}^{1,2})$ by triples $(\tau(a), \tau(b), \tau(c))$, the iteration under R maps such an element on $(R\tau(c), R\tau(a), R\tau(b))$. Now, we observe that this action has order 3 (exactly order 3 since it is not trivial). Hence, as any nontrivial automorphism of three-dimensional spaces of order 3, the action induced by R one the quotient $H^1(\Gamma, \mathbb{R}^{1,2})$ has a line of fixed points Δ, and a plane on which it acts as a rotation with angle $2\pi/3$. In particular, Δ is the unique line globally preserved by R. Hence, if the miracle imagined above occurs, E, which is obviously R-invariant, must be equal to Δ.

The next step is the identification of Δ: if a cocycle τ represents a fixed point of R in $H^1(\Gamma, \mathbb{R}^{1,2})$, it is cohomologous to $(\tau + R(\tau) + R^2(\tau))/3$. Hence, elements of $H^1(\Gamma, \mathbb{R}^{1,2})$ fixed by R are all represented by triples $(\tau(a), \tau(b), \tau(c))$ satisfying $\tau(a) = R\tau(c)$, $\tau(b) = R\tau(a)$, $\tau(c) = R\tau(b)$. The cocycle property of such a triple reduces to:

$$\tau(a) + aR\tau(a) + abR^2\tau(a) = 0,$$

i.e.

$$\tau(a) + aR\tau(a) + (aR)^2\tau(a) = 0.$$

Thus, the element aR of $\mathrm{SO}_0(1, 2)$ is of special interest. Observe that it has order 3. Indeed:

$$(aR)^3 = a(Ra)(Ra)R = ab(Rb)R^2 = abcR^3 = id.$$

It is also nontrivial, since $aR(\kappa) = \alpha$. Hence, in $\mathbf{M}^3$, it admits a timelike line of fixed points, and the orthogonal to this line of fixed points is a spacelike plane P_0. Now, we observe that the cocycle property stated above for $(\tau(a), R\tau(a), R^2\tau(a))$ means precisely that $\tau(a)$ must belong to P_0!

Hence, the line of fixed points Δ is a quotient of P_0 by elements $\tau(a)$ representing R-invariants *coboundaries*. These coboundaries form a subspace of P_0 of dimension 1, since Δ has dimension 1, and P_0 has dimension 2. We claim that these coboundaries are precisely $\tau(a) = ax - x$ where x is a fixed point of R. Indeed:

- The triple $(ax - x, bx - x, cx - x)$ is R-invariant: $R(ax - x) = Rax - Rx = bRx - x = bx - x$, and similarly, $R(bx - x) = cx - x$.
- The vector $ax - x$ belongs to P_0. Indeed:

$$\begin{aligned}&(ax - x) + aR(ax - x) + (aR)^2(ax - x)\\ &\quad = ax - x + aRax - ax + aRaRax - aRax = (aR)^3x - x = 0.\end{aligned}$$

Now, we observe that the $ax - x$ are exactly the vectors in P_0 orthogonal to α. In other words, the R-fixed triples $(\tau(a), R\tau(a), R^2\tau(a))$ representing elements in E are all coboundaries. It means that the intersection between E and Δ is reduced to $\{0\}$. In particular, $E \neq \Delta$: as observed previously, it implies that E is reduced to $\{0\}$. □

Remark 4.23. The proof above highly relies on the specific symmetric properties of the group. The identification of $\mathcal{T}^+$ for geometrically finite Kleinian groups remains an interesting challenge, even in dimension $2+1$.

4.4.2. Convex cocompact Kleinian groups

We first recall some well-known facts on Kleinian group. See e.g. [17]. Let Γ be a such a Kleinian group, i.e. a finitely generated (nonelementary) discrete subgroup of $\mathrm{SO}_0(1, n-1)$. Let $\hat{\Lambda}$ be its limit set in the conformal sphere $\mathcal{S} \approx \partial\mathbb{H}^{n-1}$, and let $C(\hat{\Lambda})$ be the convex hull of $\hat{\Lambda}$ in $\mathbb{H}^{n-1}$. This is the minimal Γ-invariant closed convex subset of $\mathbb{H}^{n-1}$.

Definition 4.24. Γ is *geometrically finite* if, for any $\epsilon > 0$, the quotient of the ϵ-neighborhood of $C(\hat{\Lambda})$ by Γ has finite volume. If moreover this quotient is compact, then Γ is *convex cocompact*.

The main aspect of convex cocompact groups we will use is the *hyperbolic* character of their dynamics around the limit sets. Namely (see [20], Section 9), there is a finite symmetric generating set G for Γ of loxodromic elements $\gamma_1, \ldots, \gamma_k$ such that:

(i) For some fixed round metric on $\mathcal{S}$ in the natural conformal class, if U_i denotes the (open) domain of the sphere where g_i is expanding, then the union of the U_i covers $\hat{\Lambda}$.
(ii) There is an uniform N such that, for any x_0 in $\hat{\Lambda}$, and for any pair of sequences $\gamma_{i_0}, \gamma_{i_1}, \ldots$ and $\gamma_{j_0}, \gamma_{j_1}, \ldots$ satisfying $x_{n+1} = \gamma_{i_n} x_n \in U_{i_{n+1}}$ and $x'_{n+1} = \gamma_{j_n} x'_n \in U_{j_{n+1}}$, then $(\gamma_{j_n} \ldots \gamma_{j_0})(\gamma_{i_n} \ldots \gamma_{i_0})^{-1}$ is equal to a product $\gamma_{k_l} \ldots \gamma_{k_0}$ with $l \leq N$.

Actually, Sullivan in [20] states the hyperbolic property of convex cocompact Kleinian groups only in the three-dimensional case, but his proof applies for the general case in any dimension: indeed, the proof in [20] relies on the fact that the limit set admits only *conical* limit points, and this fact remains true in any dimension (in fact, it is another equivalent definition of convex cocompact groups, see [17]). Anyway, any reader acquainted with dynamical systems theory will recognize this hyperbolic property on $\hat{\Lambda}$ as the hyperbolic property of the geodesic flow on $\Gamma \setminus \mathbb{H}^n$ around the compact invariant set formed by the geodesics lying entirely in $\Gamma \setminus C(\hat{\Lambda})$.

Observe that this hyperbolic property extends directly to the action of $\Gamma \subset \mathrm{SO}_0(1, n-1) \subset \mathrm{Isom}(\mathbf{M}^n)$ on $\mathcal{J}$, where the compact invariant subset is $\hat{\Lambda} \times \{0\}$ (here, the action on $\mathbf{M}^n$ under consideration is the action associated to the trivial cocycle). Indeed, the property (i) follows from Proposition 4.2 and the expanding property of Γ on $\mathcal{S}$, and the property (ii) follows directly from its version on $\mathcal{S}$.

Hence, and as observed in the last remark p. 259 of [20], Theorem II of [20] can be applied: the action of Γ on $\hat{\Lambda} \times \{0\}$ is structurally stable. In other words, for any action of Γ on $\mathcal{J}C^0$-near the initial hyperbolic action, there is a compact invariant subset, on which the action restricts as an action topologically conjugate to the restriction of Γ on $\hat{\Lambda} \times \{0\}$. This is true in particular for actions associated to nontrivial cocycles in $Z^1(\Gamma, \mathbb{R}^{1,n-1})$ which are sufficiently small, i.e., for which the $\tau(\gamma_i)$ $(1 \leq i \leq k)$ have small N-norm. These small cocycles correspond to actions on $\mathcal{J}$ preserving a compact subset; therefore, they belong to $\mathcal{T}^+ \cap -\mathcal{T}^+$. Since $\mathcal{T}^+$ is a cone.

Theorem 4.25. *For convex cocompact Kleinian groups, the convex cone $\mathcal{T}^+$ is the whole $H^1(\Gamma, \mathbb{R}^{1,n-1})$.*

5. Around Bieberbach's theorem

5.1. Non-discrete linear parts and Auslander's theorem

We will use the following theorem, refinement by Carrière and Dal'bo of a theorem by Auslander (see [9], Theorem 1.2.1).

Theorem 5.1. *Let Γ be a discrete subgroup of* $\mathrm{Aff}(n, \mathbb{R})$. *Then, the identity component G_0 of the closure of $L(\Gamma)$ in $GL(n, \mathbb{R})$ is nilpotent. In particular, $\Gamma_{nd} = L^{-1}(G_0 \cap L(\Gamma))$ is nilpotent.*

This theorem fits perfectly with (see [9], Proposition 1.2.2).

Theorem 5.2. *Let Γ be a nilpotent group of affine transformations of $\mathbb{R}^n$. Then, there exist a maximal Γ-invariant affine subspace* **U** *of $\mathbb{R}^n$ such that the restriction of the action of Γ to* **U** *is unipotent. Moreover, this maximal unipotent affine subspace is unique.*

As an application of these theorems, it is easy, for example, to recover the following version of Bieberbach's Theorem:

If Γ is discrete group of isometries of the Euclidean space of dimension n, it contains a finite index free abelian subgroup of finite rank.

The proof goes as follows: if $L(\Gamma) \subset \mathrm{SO}(n)$ is discrete, it is finite; the kernel of $L_{|\Gamma}$ is a finite index subgroup of translations. If $L(\Gamma)$ is not discrete, we consider the unique maximal unipotent affine subspace associated to Γ_{nd}: Γ_{nd} is a finite index subgroup of Γ, and the kernel of the restriction morphism $\Gamma \to \mathrm{Isom}(\mathbf{U})$ is a finite index subgroup of Γ_{nd}. The proof is completed by the observation that unipotent elements of $\mathrm{Isom}(\mathbf{U})$ are translations.

The most famous version of Bieberbach's theorem is maybe the following corollary:

If Γ is a crystallographic group, i.e. a discrete group of isometries such that $\mathbb{R}^n/\Gamma$ is compact, then it contains a finite index subgroup of translations.

This complement arises from a homological argument: since $\mathbb{R}^n/\Gamma$ is compact, any finite index subgroup has a non-trivial Betti-number b_n, but since there is a finite index subgroup of Γ acting as translations on **U**, this finite index subgroup has trivial Betti numbers b_i for $i \geq \dim(\mathbf{U})$.

5.2. Twisted product by Euclidean manifolds

Truly speaking, it is easy to provide a more precise version of Bieberbach's theorem in the non-compact case, but requiring a new notion: let M be any manifold, and denote by Γ its fundamental group. Let N be any flat Euclidean manifold, and $r : \Gamma \to \mathrm{Isom}(N)$ be any morphism. It is well-known how to produce from such a data a locally trivial bundle $q : B \to M$ with fibers diffeomorphic to N: if $\tilde{M}$ is the universal covering of M, consider the diagonal action $(\tilde{x}, y) \mapsto (\gamma\tilde{x}, r(\gamma)(y))$ of Γ on the product $\tilde{M} \times N$. Since the action of Γ on the first component $\tilde{M}$ is free and properly discontinuous, the same is true for its action on $\tilde{M} \times N$. Denote by B the quotient space, and $q : B \to M$ the map induced by the first projection map. Clearly, q is a locally trivial fibration, with fiber N as claimed above. It is called the *suspension over M with monodromy r*.

Now, if M is a flat Euclidean manifold, the product metric on $\tilde{M} \times N$ is locally Euclidean (since N is assumed here locally Euclidean) and is preserved by the Γ-action defined above. Therefore, the quotient space B inherits itself a locally Euclidean structure, canonically defined from the initial locally Euclidean structures on M and N. B, equipped with this Euclidean structure, is called the *twisted product of M by N with monodromy r*.

The argument in the preceding section actually show that *any locally Euclidean manifold is a twisted product of a flat torus* $\Gamma\backslash\mathbf{U}$ *by a Euclidean linear space* $\mathbb{R}^n/\mathbf{U}$ *with holonomy in* $\mathrm{SO}(\mathbb{R}^n/\mathbf{U})$.

It should be clear to the reader that a similar procedure can be defined when M is a flat Lorentzian manifold; the result is then a canonical flat Lorentzian structure on B, which is called once more *the twisted product of M by the Euclidean manifold N, with monodromy r*. When N is an Euclidean linear space and r takes value in linear isometries (i.e., $r(\Gamma)$ admits a global fixed point in N), then we say that B is a *linear twisted product over M*.

Remark 5.3. If $t : M \to \mathbb{R}$ is a proper time function expressing M as a GH space–time, the composition $t \circ q$ satisfies the same properties. Hence, *the twisted products by Euclidean manifolds of flat GH space–times are still flat GH space–times*. The inverse statement, namely the fact that M is GH as soon as B is, even if far from obvious now, follows from Theorem 1.1.

In the same vein, twisted products over Cauchy-complete GH space–times are still Cauchy-complete, but, of course, such a procedure preserves Cauchy-compactness if and only if the Euclidean manifold N is closed (i.e. finitely covered by a flat torus).

Remark 5.4. Since here we don't worry about finite index phenomena, the Bieberbach's theorem as stated above gives a completely satisfactory description of Euclidean manifolds, and their isometry groups $\mathrm{Isom}(N)$ are easily described. For example, when N is a flat torus, up to finite index, the representation $r : \Gamma \to \mathrm{Isom}(N)$ can be assumed as taking value in translations on N.

We should also point out that all twisted products we consider in this work are either linear twisted products, or twisted products by flat tori.

The following lemma will be useful to recognize twisted products, and its proof should be obvious to the reader.

Lemma 5.5. *Let B be a flat space–time, quotient of an open subset* Ω *of* $\mathbf{M}^n$ *by a group of isometries acting freely and properly discontinuously. Assume that* Γ *preserves a timelike affine subspace* $\mathbf{U}$, *and that* Ω *is invariant by translations by vectors contained in the orthogonal linear space* $\mathbf{U}^{\perp}$. *Then, the quotient of* $\mathbf{U} \cap \Omega$ *by* Γ *is a flat space–time M, and B is the linear twisted product of M by the Euclidean space* $\mathbf{U}^{\perp}$.

6. Classification of isometries

Let g be an isometry of $\mathbf{M}^n$. We denote by $L(g)$ its linear part, p the element $L(g) - id$ of $\mathrm{SO}_0(1, n-1)$, and τ the translation part: $g(x) = L(g)(x) + \tau$.

Observe that:

$$(*) \quad \langle x|p(y)\rangle + \langle p(x)|y\rangle + \langle p(x)|p(y)\rangle = 0.$$

In particular, if we denote by I the image of p, the kernel of p is the orthogonal $I^{\perp}$.

Moreover, τ is defined modulo I, since it can be modified through conjugacies by translations.

Up to conjugacy, every isometry is of the following form:

- Elliptic: $L(g)$ preserves some Euclidean norm on $\mathbf{M}^n$. Then, since $\mathbf{M}^n = I \oplus I^{\perp}$, τ can be selected in $I^{\perp}$, i.e., fixed by $L(g)$. Observe that in this case, I is spacelike, except if R is the identity map. Indeed, $L(g)$ considered as an isometry of $\mathbf{H}^{n-1}$ has a fixed point there, proving that $I^{\perp}$ contains a timelike element. The claim follows.
- Hyperbolic: of the form:

$$\begin{pmatrix} ch(\zeta) & sh(\zeta) & 0 & 0 & \dots & 0 \\ sh(\zeta) & ch(\zeta) & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

  In this case, τ can be assumed in $I^{\perp}$.
- Unipotent: p is nilpotent. There is a trichotomy on this case, depending on the position of τ with respect to I and $I^{\perp}$ (see below).
- Loxodromic: of the form $R \circ A$, where R is elliptic, A hyperbolic, and $R \circ A = A \circ R$. Once more, τ should be assumed fixed by $L(g)$.
- Parabolic: of the form $R \circ A$, where R is elliptic, A unipotent, and $R \circ A = A \circ R$.

We have to understand better the parabolic case; in particular:

6.1. Unipotent linear parts

Observe that in this case, $J = I \cap I^{\perp}$ is a nontrivial isotropic space, therefore, it is generated by a single isotropic element v_0. A straightforward analysis shows that $L(g)$ is represented, in some (non-orthogonal) basis $(e_1, e_2, \dots, e_n)$, by the matrix:

$$\begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & 1 & 0 \\ 0 & \dots & \dots & \dots & \dots & 1 \end{pmatrix}.$$

Moreover, the basis is such that J is spanned by the first element e_1, I by (e_1, e_2), and $J^{\perp}$ by $(e_1, e_2, \dots, e_{n-1})$.

For the translation vector τ, we have three cases to consider:

- either it belongs to *I*, in which case we can assume it to be 0; we call this case the *linear case*;
- either it belongs to $J^{\perp} \setminus I$, in which case we can assume that it belongs to $I^{\perp}$, since $J^{\perp} = I + I^{\perp}$: we call it the *tangent case*;
- either it does not belong to $J^{\perp}$: we call this case the *transverse case*.

Remark 6.1. In the three-dimensional case, we have the equality $I = J^{\perp}$, hence, the tangent case does not occur.

6.2. Parabolic linear parts

We consider here the case $L(g) = R \circ (id + p) = (id + p) \circ R$ with *p* nilpotent and *R* a non-trivial elliptic element of $SO_0(1, n - 1)$.

The preceding section characterizes *p*. We observe that *I*, *J* and their orthogonals must be *R*-invariant. Since *R* belongs to the orthochronous component, $R(e_1) = e_1$. Being elliptic, its restriction to $I = \langle e_1, e_2 \rangle$ is the identity map.

Therefore, $R \circ p = p$. Denote by $\mathcal{I}$ the image of $R - id$; fixed points of *R* are the elements of $\mathcal{I}^{\perp}$. The claim above implies the inclusions $I \subset \mathcal{I}^{\perp}$, $\mathcal{I} \subset I^{\perp}$. Observe also that the orthogonal of $J^{\perp}$ by any *R*-invariant Euclidean norm is a *R*-invariant line: it is contained in the fixed point set of *R*, i.e., in $\mathcal{I}^{\perp}$. In other words, e_n can be selected *R*-invariant.

Now, we observe that τ is defined modulo $\mathrm{Im}(R - id + p)$. Any element w of $\mathbf{M}^n$ is a sum $w = w_1 + w_2$ with w_1 in $\mathcal{I}$ and w_2 in $\mathcal{I}^{\perp}$. Then, $(R - id + p)(w_1) = (R - id)(w_1)$ since $p(\mathcal{I}) = 0$, and $(R - id + p)(w_2) = p(w_2)$. We deduce that $\mathrm{Im}(R - id + p)$ contains *I* and $\mathcal{I}$: the first claim because $\mathcal{I}^{\perp}$ contains $I + \langle v_n \rangle$, and the second claim because $(R - id)\mathcal{I} = \mathcal{I}$. The reverse inclusions being obvious ($p(w_2) \in I$ and $(R - id)(w_1) \in \mathcal{I}$), we obtain that $\mathrm{Im}(R - id + p) = I \oplus \mathcal{I}$, the sum being direct and orthogonal.

Hence, τ can be assumed belonging to $\mathcal{I}^{\perp}$, i.e., fixed by *R*. Inside $\mathcal{I}^{\perp}$, we can select τ modulo *I*: in particular, it can be assumed orthogonal to e_2.

Once more, we distinguish three cases:

- the *linear case* if τ belongs to *I*;
- the *tangent case* if τ belongs to $J^{\perp} \setminus I$;
- the *transverse case* if τ does not belong to $J^{\perp}$.

Observe that a parabolic element is nontransverse if and only if it preserves every affine hyperplane with direction $J^{\perp}$.

7. Achronal domains of isometries

For any isometry *g*, we define its achronal domain Ω_g by:

$$\Omega_g = \{x \in \mathbf{M}^n / \forall q \in \mathbf{Z},\ |g^q(x) - x|^2 > 0\}.$$

It is the open set formed by the elements which are not causally related to any g-iterates of themselves. We describe Ω_g in every conjugacy class:

7.1. *Elliptic case*

It is the case $gx = Rx + \tau$, where R is a rotation fixing τ. Hence:

$$|g^q x - x|^2 = |(R^q - id)(x) + q\tau|^2 = |(R^q - id)(x)|^2 + q^2|\tau|^2$$

since τ is orthogonal to I which contains $|(R^q - id)(x)|^2$. Remember also that I is spacelike. Hence, we have three cases:

(1) *τ is timelike*. Then, $\Omega_g = \emptyset$ (Indeed, the term $q^2|\tau|^2$ is the leading term since $(R^q - id)(x)$ is uniformly bounded).
(2) *τ is spacelike*. Then, $\Omega_g = \mathbf{M}^n$.
(3) *τ is lightlike*. Then, the complement of Ω_g is the subspace formed by R-periodic points, i.e., the R^n-fixed points. This case contains a very special one, the linear case, where τ is 0.

7.2. *Loxodromic elements*

It is more suitable to write elements of $\mathbf{M}^n$ in the form (x, y, v) with x, y in $\mathbb{R}$ and v in $\mathbb{R}^{n-2}$, so that the Lorentzian form Q is expressed by:

$$Q = xy + \|v\|^2,$$

where $\| \|^2$ denotes the usual Euclidean norm on $\mathbb{R}^{n-2}$. Then, the action of of $L(g)$ is defined by:

$$L(g)(x, y, v) = (\lambda x, \lambda^{-1} y, Rv),$$

where $\lambda = e^{\zeta/2}$ is a positive real number, and R a rotation of $\mathbb{R}^{n-2}$. Observe also that τ can be assumed belonging to $\mathbb{R}^{n-2}$. Then:

$$\begin{aligned} |g^q u - u|^2 &= q^2|\tau|^2 + (\lambda^q - 1)(\lambda^{-q} - 1)xy + \|(R^q - id)(v)\|^2 \\ &= q^2|\tau|^2 - 4\,\mathrm{sh}^2(q\zeta)xy + \|(R^q - id)(v)\|^2. \end{aligned}$$

The leading term $-4\mathrm{sh}^2(q\zeta)xy$ ensures:

$$\Omega_g = \left\{ \frac{(x, y, v)}{xy} < 0 \right\}.$$

7.3. Parabolic elements

When p is the nilpotent part of a unipotent element $id + p$ of $SO_0(1, n-1)$, we have:

$$(id + p)^k = id + kp + \tfrac{1}{2}(k(k-1))p^2.$$

When classifying parabolic elements, we proved that the elliptic part satisfies $R \circ p = p$, and that the translation part can be assumed fixed by R and in $v_1^{\perp}$. It follows:

$$\begin{aligned} g^q(x) &= R^q(id + p)^q(x) + q\tau + \sum_{k=0}^{q-1} kp(\tau) + \sum_{k=0}^{q-1} \frac{k(k-1)}{2} p^2(\tau) \\ &= R^q x + qp(x) + \frac{q(q-1)}{2} p^2(x) + q\tau + \frac{q(q-1)}{2} p(\tau) \\ &\quad + \frac{q(q-1)(q-2)}{6} p^2(\tau). \end{aligned}$$

Hence, x belongs to Ω_g if and only if for every q, $T(q)$ is positive, where:

$$\begin{aligned} T(q) = \bigg| &\frac{R^q x - x}{q} + p(x) + \frac{(q-1)}{2} p^2(x) + \tau + \frac{(q-1)}{2} p(\tau) \\ &+ \frac{(q-1)(q-2)}{6} p^2(\tau)\bigg|^2. \end{aligned}$$

The linear case $\tau = 0$. The remaining terms all belong to $J^{\perp}$, and $p^2(x)$ belongs to J; moreover, $p(x)$ belongs to I which is orthogonal to $(R^q x - x)/q$ since this term belongs to $\mathcal{I}$. Hence, $T(q)$ is equal to $|(R^q x - x)/q|^2 + |p(x)|^2$ and is therefore nonnegative. It vanish if and only if $|p(x)|^2 = 0$, i.e., $x \in J^{\perp}$, and $|(R^q x - x)/q|^2 = 0$, i.e. $x = R^q x$ since $\mathcal{I}$ is spacelike.

In other words, the complement of Ω_g is the set of R^n-fixed points in the hyperplane $J^{\perp}$.

The tangent case $\tau \in J^{\perp} \setminus I$. Then, $p^2(\tau) = 0$. Observe that all the terms in $T(q)$ belong to $J^{\perp}$, and that $p^2(x)$, $p(\tau)$ both belong to J. Thus, the expression of $T(q)$ reduces to:

$$\left| \frac{R^q x - x}{q} + p(x) + \tau \right|^2.$$

The term $(R^q x - x)/q$ belongs to $\mathcal{I}$, and $p(x) + \tau$ belongs to $\mathcal{I}^{\perp}$. Therefore, $T(q)$ is the sum of the nonnegative terms $|(R^q x - x)/q|^2$ and $|\tau + p(x)|^2$. Moreover, since τ is assumed *not* in I, the second term never vanishes.

We conclude that in the tangent case, Ω_g is the entire $\mathbf{M}^n$.

The transverse case $\tau \in \mathbf{M}^n \setminus J^{\perp}$. In this case, $p^2(\tau)$ is not 0. We develop $T(q)$, and seek for the leading terms: since $p^2(\tau)$ is isotropic and orthogonal to $p(\tau)$, $p(x)$, the leading terms are the terms in q^2:

$$(\tfrac{1}{4}(q-1)^2)|p(\tau)|^2 + 2(\tfrac{1}{6}(q-1)(q-2))\langle\tau|p^2(\tau)\rangle.$$

According to (∗):

$$\langle\tau|p^2(\tau)\rangle + |p(\tau)|^2 + \langle p(\tau)|p^2(\tau)\rangle = 0.$$

Hence, keeping in mind $\langle p(\tau)|p^2(\tau)\rangle = 0$, the leading term is:

$$(\tfrac{1}{12}(3(q-1)^2 - 4(q-1)(q-2)))|p(\tau)|^2 = (\tfrac{1}{12}((q-1)(5-q)))|p(\tau)|^2.$$

Since $|p(\tau)|^2 > 0$, we obtain that $T(q)$ is negative for big q:

Conclusion: Ω_g is empty.

7.4. Visibility of one point from another

Points of the achronal domain studied above are points which cannot observe any g-iterate of themselves. We wonder now, a base point x_0 being fixed, how many g-iterates of a point x are causally related to x_0? For our purpose, we just need to consider the case where g is parabolic.

Lemma 7.1. *Let g be a parabolic isometry. Let x_0 be any point of $\mathbf{M}^n$. Then, there exist at least one point x in $\mathbf{M}^n$ admitting infinitely many g-iterates in the future or in the past of x_0.*

Proof. We want to prove the existence of x such that for infinitely q, the norm $|g^q(x) - x_0|^2$ is negative. We have:

$$|g^q(x) - x_0|^2 = |x|^2 + |x_0|^2 - 2\langle g^q(x)|x_0\rangle.$$

In the transverse case, the leading term in q for $\langle g^q(x)|x_0\rangle = \langle g^q(x_0)|x\rangle$ is:

$$\langle(\tfrac{1}{6}(q(q-1)(q-2)))p^2(\tau)|x\rangle.$$

When x is not orthogonal to $p^2(\tau)$, this quantity is positive with arbitrarily big values for infinitely many q. Thus, such a x admits infinitely g-iterates causally related to x_0: the lemma is proven in this case.

In the linear or tangent cases, the leading term for $\langle g^q(x)|x_0\rangle$ is:

$$\tfrac{1}{2}q(q-1)\langle p^2(x) + p(\tau)|x_0\rangle.$$

Therefore, the lemma is proven in this case by taking x for which $\langle p^2(x) + p(\tau)|x_0\rangle$ is positive. □

8. Spacelike hypersurfaces and achronal domains

From now, we consider a flat space–time M of dimension n, and an isometric immersion $f : S \to M$ of a *complete* Riemannian hypersurface. Let $\tilde{M}$ be the universal covering of M, $\mathcal{D} : \tilde{M} \to \mathbf{M}^n$ the developing map, Γ the fundamental group of S, and $\tilde{f} : \tilde{S} \to \tilde{M}$ a lifting of f. There is an action of Γ on $\tilde{M}$, a priori non-injective, such that $\tilde{f}$ is an equivariant map.

Let ρ be the holonomy morphism of M restricted to Γ: it is a morphism $\rho : \Gamma \to \mathrm{Isom}(\mathbf{M}^n)$.

We begin with an easy, but fundamental observation (see, for example, [16], or [18]).

Proposition 8.1. *Let Δ be a timelike direction in $\mathbf{M}^n$. Let $Q(\Delta)$ be the quotient linear space $\mathbf{M}^n/\Delta$, and $\pi : \mathbf{M}^n \to Q(\Delta)$ the quotient map. Equip $Q(\Delta)$ with the metric for which π restricted to every spacelike hyperplane orthogonal to Δ is an isometry. Then, $\pi \circ \mathcal{D} \circ \tilde{f}$ is a homeomorphism.*

Proof. The map $\pi \circ \mathcal{D} \circ \tilde{f}$ is distance increasing, and thus, a local homeomorphism. Since $\tilde{S}$ is complete, it has moreover the lifting property. Therefore, $\pi \circ \mathcal{D} \circ \tilde{f}$ is a covering map. The proposition follows since $Q(\Delta)$ is simply connected. □

Corollary 8.2. *The maps f and $\mathcal{D} \circ \tilde{f}$ are embeddings. The natural morphism $\Gamma \subset \pi_1(M)$ and the morphism ρ are injective. Moreover, $\mathcal{D} \circ \tilde{f}(\tilde{S})$ intersects every timelike geodesic.*

From now, we identify $\tilde{S}$ with its image under $\mathcal{D} \circ \tilde{f}$, and Γ with its image under ρ.

Corollary 8.3. *Let P_0 be a lightlike affine hyperplane of $\mathbf{M}^n$, with direction the orthogonal of a lightlike direction Δ_0. Then, either $\tilde{S}$ does not intersect P_0, or it intersects every affine line with direction Δ_0 contained in P_0.*

Proof. The proof is completely similar to the proof of 8.1; the restriction to $S \cap P_0$ of the projection onto P_0/Δ_0 is an isometry. □

Corollary 8.4. *If Γ preserves a spacelike affine subspace* $\mathbf{U}$, *then Γ contains as a finite index subgroup a free abelian group of finite rank.*

Proof. Apply Proposition 8.1 with $\Delta \subset \mathbf{U}^{\perp}$. The action of Γ on $\pi(\mathbf{U})$ has to be free and properly discontinuous, and the same is true for the action on $\mathbf{U}$. But this action is isometric for the induced Euclidean metric on $\mathbf{U}$; the lemma then follows from Bieberbach's Theorem. □

As an immediate corollary of Proposition 8.1, we obtain that $\tilde{S}$ is the graph of a contracting map $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. Inversely, it is worth to extend the definition of spacelike hypersurfaces to (local) graphs of contracting maps from (Euclidean) $\mathbb{R}^{n-1}$ into (Euclidean) $\mathbb{R}$. In other words, it is not necessary to consider spacelike hypersurfaces as smooth objects, Lipschitz regularity is enough. This low regularity is sufficient to define a (non-Riemannian) length metric on the spacelike hypersurface, since Lipschitz maps are differentiable almost everywhere: if $c : t \mapsto \mathbb{R}^{n-1}$ is a C^1-curve in $\mathbb{R}^{n-1}$, define the length of $\varphi \circ c$ as the Lebesgue integral of $\sqrt{1 - \varphi'(c(t))^2}\|c'(t)\|$.

Another corollary is the following global property: spacelike hypersurfaces are achronal closed set, meaning that if x and y are two distinct points in $\tilde{S}$, one cannot be contained in the past of future cone of the other, i.e., we have $|x-y|^2>0$. Since $\tilde{S}$ is γ-invariant, it follows that it is contained in Ω_γ for every element γ of Γ. Actually, thanks to Corollary 8.3, we have a quite better statement.

Definition 8.5. For every isometry g of $\mathbf{M}^n$. We define an open set $U(g)$ in the following way:

- $U(g)=\Omega_g$ if g is loxodromic, or spacelike elliptic, or non-linear parabolic.
- $U(g)=\emptyset$ if g is nonspacelike elliptic.
- $U(g)=\mathbf{M}^n\setminus J^\perp$ if g is linear parabolic, where J is the lightlike affine line of fixed points of g.

Lemma 8.6. *The complete spacelike hypersurface $\tilde{S}$ is contained in the interior of $U(\Gamma)=\bigcap_{\gamma\in\Gamma}U(\gamma)$.*

Proof. First, we prove the inclusion $\tilde{S}\subset U(\gamma)$ for every nontrivial element γ of Γ. We observed previously this inclusion when $U(\gamma)=\Omega_\gamma$; thus, we have only two cases to consider:

- *The linear parabolic case*. Since $\tilde{S}\subset\Omega_\gamma$, $\tilde{S}$ cannot intersect the lightlike line of fixed points J. According to Corollary 8.3, it cannot intersect $J^\perp$.
- *The nonspacelike elliptic case*. In this case, $L(\gamma)$ admits a timelike line of fixed points. According to corollary 8.2, this line intersects $\tilde{S}$ at some point x. Then, $\gamma x=x+\tau$: this is impossible since τ is nonspacelike and that $\tilde{S}$ is achronal.

Consider now a point x in $\tilde{S}$, and B a small convex neighborhood of x such that lightlike geodesic segment contained in B intersects $\tilde{S}$. Then, for every γ, the complement of $U(\gamma)$ is a union of lightlike geodesic which cannot intersect $\tilde{S}$, and thus, cannot intersect B. It proves that $\tilde{S}$ is contained in the interior of $U(\Gamma)$. □

Definition 8.7. The interior of $U(\Gamma)$ is denoted by $\Omega(\Gamma)$ and called the achronal domain of Γ. The connected component of the interior of $U(\Gamma)$ containing $\tilde{S}$ is denoted by $\Omega(\tilde{S})$.

Remark 8.8. For any isometry g, and for every nonzero integer n we have $U(g^n)=U(g)$. Therefore, if Γ' is a finite index subgroup of Γ, the equality $U(\Gamma')=U(\Gamma)$ holds. Thanks to this remark, we can replace Γ by any finite index subgroup.

Remark 8.9. From now, we will assume that the group Γ does not contain nonspacelike elliptic and transverse parabolic elements, since we have shown that in this case Γ cannot be the holonomy group of a Cauchy-complete GH space–time.

Proposition 8.10. $\Omega(\tilde{S})$ is a convex open set. It is globally hyperbolic. The action of Γ on it is free and achronal.

Proof. The convexity follows from the convexity of connected components of every $U(\gamma)$. The action of Γ on $\Omega(\tilde{S})$ is obviously free and achronal. The global hyperbolicity follows from the following criterion (see [4]): a space–time is *distinguished* if for every pair $x\neq y$, the causal futures *and* causal pasts of x, y are distinct. Now, a distinguished space–time is globally hyperbolic if for any pair of points x, y, the intersection between the causal past of x and the causal future of y is empty or compact.

The open set $\Omega(\Gamma)$ is obviously distinguished, we can thus apply the criterion above: let x, y be two points in $\Omega(\Gamma)$, with x in the causal future of y. Then they are points x', y' near x, y in $\Omega(\Gamma)$, and such that x is in the past of x' and y in the future of y'—we mean the causality relation in $\mathbf{M}^n$. Then, $F(y) \cap P(x)$ is contained in the interior of $F(y') \cap P(x')$, and this interior is contained in $\Omega(\Gamma)$ since $F(y') \cap P(x')$ is contained in every $U(\gamma)$. □

From now, we assume that M is a Cauchy-complete globally hyperbolic space–time admitting S as a complete Cauchy hypersurface of M. The following observation is well-known.

Proposition 8.11. *M is homeomorphic to $S \times \mathbb{R}$, where every $S \times \{*\}$ are Cauchy hypersurfaces. Any nonspacelike geodesic in $\tilde{M}$ intersect $\tilde{S}$.*

Proof. See [4]. □

Proposition 8.12. *The developing map $\mathcal{D} : \tilde{M} \to \mathbf{M}^n$ is injective, with image contained in $\Omega(\tilde{S})$.*

Proof. Fix a timelike direction Δ in $\mathbf{M}^n$. For any element $\tilde{x}$ of $\tilde{M}$, let $\delta(\tilde{x})$ be the line containing $\mathcal{D}(\tilde{x})$ with direction Δ, and $d(\tilde{x})$ the connected component of $\mathcal{D}^{-1}(\delta(\tilde{x}))$ containing $\tilde{x}$. Then, $d(\tilde{x})$ is a timelike geodesic in $\tilde{M}$ and must therefore intersect $\tilde{S}$ at a single point $p(\tilde{x})$.

If $\mathcal{D}(\tilde{x}) = \mathcal{D}(\tilde{y})$, then $\delta(\tilde{x}) = \delta(\tilde{y}) = \delta$ and $\mathcal{D}(p(\tilde{x})) = \delta \cap \mathcal{D}(\tilde{S}) = \mathcal{D}(p(\tilde{y}))$. Hence, according to Corollary 8.2, $p(\tilde{x}) = p(\tilde{y})$, and thus, $d(\tilde{x}) = d(\tilde{y})$. But the restriction of $\mathcal{D}$ to $d(\tilde{x})$ is an local homeomorphism from a topological line into $\mathbb{R}$: it is therefore injective. We obtain $\tilde{x} = \tilde{y}$, i.e., $\mathcal{D}$ is injective.

The homeomorphism $M \approx S \times \mathbb{R}$ lifts to a homeomorphism $\tilde{M} \approx \tilde{S} \times \mathbb{R}$ where every $\tilde{S} \times \{t\}$ is a Cauchy hypersurface for $\tilde{M}$. Hence, for any t, every nonspacelike geodesic intersecting $\tilde{S}$ meets $\tilde{S} \times \{t\}$. It follows, thanks to Proposition 8.1, that every nonspacelike line of $\mathbf{M}^n$ meets $\mathcal{D}(\tilde{S} \times \{t\})$. That's all we need to reproduce the arguments used in the preceding section, leading to the conclusion that for any t, $\mathcal{D}(\tilde{S} \times \{t\})$ is contained in $\Omega(\Gamma)$. Since $\tilde{M}$ is connected, we obtain that the image of $\mathcal{D}$ is contained in $\Omega(\tilde{S})$. □

All the results above suggest a nice way towards the proof of the main theorems: if the action of Γ on $\Omega(\tilde{S})$ is proper, the quotient space $M(S) = \Gamma \setminus \Omega(\tilde{S})$ is well-defined, and any globally hyperbolic manifold containing S can be isometrically embedded in $M(S)$. Unfortunately, things are not going so nicely: indeed, for unipotent space–times, $\Omega(\tilde{S})$ can be the entire Minkowski space whereas the action of Γ on $\mathbf{M}^n$ is not proper (see Remark 9.4).

Anyway, this approach will be essentially successful after ruling out a special case including unipotent space–times: the nonloxodromic case, i.e., the case where Γ has no loxodromic element: this is the topic of the next section.

Remark 8.13. Actually, when M is maximal globally hyperbolic, $\mathcal{C}(\tilde{S})$, the image of $\mathcal{D}$, is the so-called Cauchy domain, or domain of dependence of the spacelike hypersurface S as defined in [16,2] or [8]: it is the open set formed by points such that every lightlike geodesic through the point intersects S. When the initial hypersurface S is closed, $\Omega(\tilde{S})$ is the Cauchy domain of $\tilde{S}$ (see remark 11.1); but this is not true in the general case.

9. The nonloxodromic case

This is the case where Γ has no loxodromic element.

Proposition 9.1. *If Γ does not contain loxodromic elements, it is elementary; i.e., up to finite index, its linear part is either contained in* $\mathrm{SO}(n-1)$ (*up to conjugacy*), *or its linear part stabilizes an isotropic vector*.

Proof. If Γ has no loxodromic elements, according to [6], the same is true for its Zariski closure G. Since G has a finite number of connected components, we can assume, restricting Γ to a finite index subgroup if necessary, that Γ is contained in the identity component G_0. If G_0 admits a parabolic element g_1 with fixed point x in $\mathbf{S}^{n-2} = \partial\mathbf{H}^{n-1}$, then x is fixed by every parabolic element of G_0: indeed, if g_2 is another parabolic element with fixed point $x_2 \neq x_1$, for any small neighborhood U of x_2, $g_1^k(U)$ is arbitrarily near x_1, i.e. far from x_2 for big k. In particular, $g_1^k(U)$ does not contain x_2. Therefore, for big l, $g_2^l g_1^k(U)$ is arbitrarily near x_2. In particular, if k and l are sufficiently big, the closure of $g_2^l g_1^k(U)$ is contained in U, showing that $g_2^l g_1$ is loxodromic.

Hence, if G_0 admits a parabolic element, the fixed point of this parabolic element is fixed by every element of G_0, meaning that $L(\Gamma)$ stabilizes a lightlike direction.

If not, G_0 admits only elliptic elements. Therefore, 1-parameter subgroups are all compact: it follows that G_0 is compact, hence, contained in a conjugate of $SO(n-1)$. □

According to Proposition 9.1, the nonloxodromic case decomposes in the elliptic case, and the parabolic case:

9.1. The elliptic case

There is a flat Euclidean metric on $\mathbf{M}^n$ preserved by Γ. Remember also that according to 8.6, elliptic elements of Γ are all spacelike. Hence, $U(\Gamma) = \mathbf{M}^n$, and Γ acts freely on $\mathbf{M}^n$. According to Bieberbach's Theorem, up to finite index, there is a unique maximal affine subspace $\mathbf{U}$ on which Γ acts by translations. Moreover, $\mathrm{SO}(n-1) \subset \mathrm{SO}(1, n-1)$ is the stabilizer of a point in $\mathbf{H}^{n-1}$, i.e., preserves a timelike direction Δ_0: it follows that $\mathbf{U}$ is timelike. According to Lemma 5.5, $M(S)$ is thus a linear twisted product over a translation space–time.

9.2. The parabolic case

In this case, up to finite index, $L(\Gamma)$ preserves an isotropic vector v_0 and Γ contains a parabolic element γ_1. Denote by P_0 the orthogonal $v_0^{\perp}$, by $\Delta_0 = P_0^{\perp}$ the direction containing v_0, and by $\overline{P}_0$ the quotient space P_0/Δ_0. The Lorentzian quadratic form on $\mathbf{M}^n$ induces on $\overline{P}_0$ an Euclidean quadratic form $\overline{q}_0$ which is preserved by the natural action of $L(\Gamma)$. The quotient space $Q_0 = \mathbf{M}^n/\Delta_0$ is a $(n-1)$-dimensional space foliated by affine planes with direction $\overline{P}_0$. We denote by $\pi_0 : \mathbf{M}^n \to Q_0$ the quotient map. Finally, we denote by $G(v_0)$ the group of isometries of $\mathbf{M}^n$ preserving v_0. The group $G(v_0)$ has an induced affine action on Q_0: there is a representation $\alpha : G(v_0) \to \mathrm{Aff}(Q_0)$ for which π_0 is equivariant. The kernel of α is generated by translations along Δ_0.

In a similar way, we consider the quotient map $\pi_1 : \mathbf{M}^n \to \mathbf{M}^n/P_0 = Q_1$: the induced action of an element g of $G(v_0)$ on $\mathbf{M}^n/P_0$ is a translation by a real number $b(g)$. There is a canonical map $f : Q_0 \to Q_1$.

We keep in mind that Γ preserves the complete spacelike hypersurface $\tilde{S}$: this has some implications:

(1) The projection $\pi_0(\tilde{S})$ is an open set of the form $f^{-1}(I)$ where I is a segment of the affine line Q_1 (cf. Corollary 8.3).
(2) $\Omega(\tilde{S})$ is $\pi_1^{-1}(I')$ where I' is a segment of Q_1 containing I. Indeed, every $U(\gamma)$ is a preimage by π_1 of a half-line in Q_1.
(3) The action of Γ on $\pi_0(\tilde{S})$ via α is free and properly discontinuous (since the restriction of π_0 to $\tilde{S}$ is a homeomorphism onto its image).

Thus, point (3) above implies that $\alpha(\Gamma)$ is a discrete subgroup of $\mathrm{Aff}(Q_0)$.

9.2.1. A coordinate system on Q_0

Select a lightlike vector v_n in $\mathbf{M}^n$ transverse to P_0 and such that $\langle v_0|v_n\rangle = 1$, and denote abusively $\overline{P}_0$ the intersection $P_0 \cap v_n^{\perp}$. Then, we decompose every element of $\mathbf{M}^n$ in $xv_0 + z + yv_n$ with z in $\overline{P}_0$. The space Q_0 can be canonically decomposed in the form $\overline{P}_0 \times \mathbb{R}$, so that the projection map π_0 is given by:

$$\pi_0(xv_0 + z + yv_n) = (z, y).$$

Now, the action of the image under α of an element g of $G(v_0)$ acts on $Q_0 \approx \overline{P}_0 \times \mathbb{R}$ has the following expression:

$$(**) \quad \alpha(g)(z, y) = (R(g)(z) + u(g) + yv(g), y + b(g)),$$

where $u(g)$, $v(g)$ belong to $\overline{P}_0$, $R(g)$ is a rotation in $\overline{P}_0$, and $b(g)$ a real number.

The action of the linear part of g on $\mathbf{M}^n \approx \mathbb{R} \oplus \overline{P}_0 \oplus \mathbb{R}$ can be expressed by:

$$g(xv_0 + z + yv_n) = (x - \langle v(g)|R(g)(z)\rangle - \tfrac{1}{2}y|v(g)|^2)v_0 + (R(g)z + yv(g)) + yv_n,$$

The translation part being given by $\mu(g)v_0 + u(g) + b(g)v_n$, where $\mu(g)$ is not characterized by $\alpha(g)$. Observe that the linear part $L(g)$ is uniquely defined by the induced isometry $\lambda(g)$ of $\overline{P}_0$, i.e., by the pair (R, v). An element g is elliptic if $\lambda(g)$ admits a fixed point in $\overline{P}_0$. If not, g is parabolic.

The action on Q_0 of purely unipotent elements of $G(v_0)$ has the following expression:

$$(**') \quad \alpha(g)(z, y) = (z + u(g) + yv(g), y + b(g)).$$

Therefore, they form a group $\mathcal{U}(v_0)$. Their action on $\mathbf{M}^n$ is:

$$\begin{aligned} g(xv_0 + z + yv_n) = {} & (x - \langle v(g)|z\rangle - \tfrac{1}{2}y|v(g)|^2 + \mu(g))v_0 \\ & + (z + yv(g) + u(g)) + (y + b(g))v_n. \end{aligned}$$

We distinguish the kernel $\mathcal{A}(v_0)$ of $b : \mathcal{U}(v_0) \to \mathbb{R}$. Every element g of $\mathcal{A}(v_0)$ is characterized by $u(g), b(g), \mu(g)$. We recognize the group $\mathcal{A}$ discussed in the introduction for the definition of unipotent space–times.

Remark 9.2. The coordinate system is subordinated to the initial choice of v_n which is defined up to an element of P_0. Select w an element of P_0, and consider the new coordinates (x', z', y'), and the new morphisms u', v', μ' defined with respect to the decomposition $\Delta_0 \oplus (P_0 \cap (v_n + w)^{\perp}) \oplus \langle v_n + w \rangle$.

- If w is a multiple av_0, then $xv_0 + z + yv_n = (x - ay)v_0 + z + y(v_n + av_0)$, showing that $u' = u$, $v' = v$ and $\mu' = \mu$.
- If w belongs to $\overline{P}_0 = P_0 \cap v_n$, then $y' = y$, $u' = u + \langle u|w \rangle v_0$, $v' = v + \langle v|w \rangle v_0$. It follows:

$$\mu'(g) = \mu(g) + \langle u(g)|w \rangle.$$

9.2.2. Triviality of b

Assume the existence of some element γ_0 of Γ for which $b(\gamma_0) \neq 0$. Then, being invariant by a non-trivial translation, the interval $I = \pi_1(\tilde{S})$ must be the whole Q_1. According to Corollary 8.3, it follows that $\tilde{S}$ intersect every fiber of π_0. Remember now that we assume here the existence of a parabolic element γ_1 in Γ, and Lemma 7.1: select the base point x_0 in $\tilde{S}$ and consider the points x of $\mathbf{M}^n$ admitting infinitely many γ_1-iterates causally related to x_0. At one hand, according to the proof of Lemma 7.1, suitable x are the points for which $\langle p^2(x) + p(\tau)|x_0 \rangle$ is positive, thus x can be selected up to Δ_0; in particular, it can be selected in $\tilde{S}$. On the other hand, since they both belong to the Γ-invariant hypersurface $\tilde{S}$, no Γ-iterate of x can be causally related to x_0. We thus obtained a contradiction.

9.2.3. The purely unipotent case

We have just proved that the morphism b is trivial, i.e., the action of Γ on Q_1 is trivial. This case cannot arise when the initial spacelike hypersurface is compact. Indeed, π_1 is then Γ-invariant, inducing a submersion from S onto $\mathbb{R}$.

Modifying the coordinate system, we can assume that 0 belongs to $I = \pi_1(\tilde{S})$. According to point (3) above, Γ acts freely and properly discontinuously on $\overline{P}_0 \times \{0\}$-observe that this action is the Euclidean action defined through $\lambda(\Gamma)$. According to Bieberbach's theorem, up to finite index, Γ acts effectively by translations on some affine subspace $\overline{\mathcal{U}}_0 \times \{0\} \subset \overline{P}_0 \times \{0\}$. In particular, every $u(\gamma)$ belongs to the direction of $\overline{\mathcal{U}}_0$. But this observation works for any $t \in I$: there is some affine subspace $\overline{\mathcal{U}}_t \times \{t\} \subset \overline{P}_0 \times \{t\}$ on which the action of $\alpha(\gamma)$ has the expression:

$$\alpha(\gamma)(z, t) = (z + u(\gamma) + tv(\gamma), t)$$

When z belongs to $\overline{\mathcal{U}}_t$. It follows that the action of every $\lambda(\gamma)$ on the linear space spanned by all the $\overline{\mathcal{U}}_t$ is unipotent; since $\overline{\mathcal{U}}_0$ is the *maximal* $\lambda(\Gamma)$-unipotent subspace, it must contain every $\overline{\mathcal{U}}_t$. Hence, every $v(\gamma)$ belongs to $\overline{\mathcal{U}}_0$: $\overline{\mathcal{U}}_t$ does not depend on t.

In the coordinate system $\mathbf{M}^n \approx \mathbb{R} \oplus \overline{P}_0 \oplus \mathbb{R}$, consider the subspace $\mathcal{U} = \mathbb{R} \oplus \overline{\mathcal{U}}_0 \oplus \mathbb{R}$. It is Γ-invariant, and the expression of the action of $L(\gamma) - id$ on $\mathcal{U}$ is:

$$(L(\gamma) - id)(x, z, y) = (-\langle v(g)|z\rangle - \tfrac{1}{2}y|v(g)|^2, z + yv(g), 0)$$

This action is unipotent. On the other hand, $\mathcal{U}$ is timelike. Thanks to Lemma 5.5, and since the result we are proving is up to linear twisted products, we can assume that $\mathcal{U}$ is the entire Minkowski space, i.e., Γ is contained in the abelian group $\mathcal{A}(v_0)$.

Since the action of Γ on $\overline{P}_0 \times \{0\}$ is effective, $u : \Gamma \to \overline{P}_0$ is injective. We can thus parameterize Γ by its u-translation vectors. Then, if E is the linear space spanned by the u-translation vectors, there is a linear map T from E into $\overline{P}_0$ such that:

$$\alpha(u)(z, y) = (z + u + yT(u), y).$$

Moreover, Γ is isomorphic to $\mathbf{Z}^k$. In particular, it is abelian:

$$\mu(\gamma) + \mu(\gamma') - \langle u(\gamma')|v(\gamma)\rangle = \mu(\gamma\gamma') = \mu(\gamma'\gamma) = \mu(\gamma') + \mu(\gamma) - \langle u(\gamma)|v(\gamma')\rangle.$$

It follows, for every u, u' in E:

$$\langle u|T(u')\rangle = \langle u'|T(u)\rangle.$$

The causality domain $U(u)$ is then the open set formed by the points (x, z, y) for which $u + yA(u) \neq 0$. The interval I' (remember point (2)) is a connected component of the set $\{y \in Q_1 \approx \mathbb{R}/u + yA(u) \neq 0\ \forall u \in \Gamma\}$.

Denote by y_-, y_+ the extremities (possibly infinite) of the interval $I = \pi_1(\tilde{S})$. Remember that we assume here $y_- < 0 < y_+$. When $y_\pm$ are both finite, we change the coordinates so that $y_- = -y_+$.

Proposition 9.3. *At least one of the extremities $y_\pm$ is finite. If they are both finite then, reducing as above to the case $y_- = -y_+$, for every u in E we have*:

$$|y_+T(u)| \leq |u|.$$

If $y_+ = +\infty$, then for every u in E:

$$\langle u|T(u)\rangle \geq -y_-|T(u)|^2.$$

If $y_- = -\infty$, then for every u in E:

$$\langle u|T(u)\rangle \leq -y_+|T(u)|^2.$$

Remark 9.4. When all elements of Γ are nonlinear unipotent isometries, $\Omega(\tilde{S})$ is the entire Minkowski space. Then, according to Proposition 9.3, I and I' are not equal.

Proof of 9.3. Consider two elements $X = (x, y, z)$ and $X' = (x', y', z')$ of $\tilde{S}$, and any element $\gamma = g_{(u,v,\mu)}$ of Γ (by definition, $v = T(u)$). We want to evaluate the norm of $\gamma^p X - X'$: the first observation is that $\mu(\gamma^p) = p\mu(\gamma) + (p(1-p))/2\langle u|v\rangle$. Therefore:

$$\gamma^p(x, y, z) = (x + p\mu(\gamma) + \tfrac{1}{2}p(1-p)\langle u|v\rangle - p\langle z|v\rangle - \tfrac{1}{2}yp^2|v|^2, y, z + pu + ypv).$$

The leading term for p increasing to infinity of the Minkowski norm of $\gamma^p X - X'$ is:

$$p^2(|u|^2 + (y + y')\langle u|v\rangle + yy'|v|^2).$$

Since no γ-iterate of X can be in the causal future of X', this term must be positive for every y, y' in I. Since at least one element of Γ is parabolic, i.e., it has a nonzero v-component, it follows immediately that I cannot be the entire real line. When $y_\pm$ are both finite, $y_- = -y_+$, the limit case $y = y_+$, $y' = y_-$ provides the required inequality for $u = u(\gamma)$. This inequality extend to any u in E since $u(\Gamma)$ is a lattice of E.

When $y_+ = +\infty$, the non-negativity of the leading term for increasing y and $y' = y_-$ proves the required inequality for $u = u(\gamma)$. Once again, the cocompactness of $u(\Gamma)$ in E completes the proof.

Finally, the symmetric case $y_- = -\infty$ admits a similar proof. □

Proposition 9.5. *The map $T : E \to \overline{P}_0$ can be extended to a linear map $\hat{T} : \overline{P}_0 \to \overline{P}_0$ satisfying the same properties, i.e.*:

- $\overline{T}$ *is symmetric*:

$$\langle u|T(u')\rangle = \langle u'|T(u)\rangle.$$

- *If* $y_- = -y_+$:

$$|y_+\hat{T}(u)| \leq |u|.$$

- *If* $y_+ = +\infty$:

$$\langle u|\hat{T}(u)\rangle \geq -y_-|\hat{T}(u)|^2.$$

- *If* $y_- = -\infty$:

$$\langle u|\hat{T}(u)\rangle \leq -y_+|\hat{T}(u)|^2.$$

Proof. Consider first the case $y_+ = +\infty$. Let F be the image of T, and $F^\perp$ the orthogonal of F. Let K be the kernel of T, and $K^\perp$ the orthogonal of K inside E. From the equalities $0 = \langle T(u)|u'\rangle = \langle u|T(u')\rangle$ for u in E and u' in K, we see that $K \subset F^\perp$. On the other hand, for u in $E \cap F^\perp$, we have $-y_-|T(v)|^2 \leq \langle u|T(u)\rangle = 0$. Therefore, $K = E \cap F^\perp$: the linear space $\overline{P}_0$ is the orthogonal sum $K^\perp \oplus F^\perp$. For u in $K^\perp$ and u' in $F^\perp$, we define $\hat{T}(u + u') = T(u)$. The linear map $\hat{T}$ has the required properties.

The proof for the case $y_- = -\infty$ is completely similar. The last case to consider is $y_- = -y_+$. We can assume without loss of generality that $y_+ = 1$. We decompose $T(u)$ as the orthogonal sum $T_0(u) + B(u)$, where $T_0(u)$ belongs to E and $B(u)$ to $E^\perp$. Let $B' : E^\perp \to E$ the dual of B: it is uniquely defined by $\langle B'(v)|u\rangle = \langle v|B(u)\rangle$ for every v in $E^\perp$ and every u in E.

By hypothesis, T_0 is symmetric and its sup-norm is less than or equal to 1. Therefore, it is diagonalizable over $\mathbb{R}$, and its eigenvalues have all absolute values less than or equal to 1. For every symmetric linear map $Z : E^\perp \to E^\perp$, define $\hat{T}_Z(u+v) = (T_0(u) + B'(v)) + (B(u) + Z(v))$. This is a symmetric extension of T, and we want to prove that Z can be selected so that $\hat{T}_Z$ has norm less than or equal to 1, i.e., that its eigenvalues have all absolute values less than or equal to 1.

But the existence of such an extension is precisely the content of Parrot's lemma (see Appendix C of [1]—we are pleased to thank D. Serre who indicated to us this theorem and this reference). □

Since $\hat{T}$ is symmetric, there is an orthonormal basis $e_1, \ldots, e_{n-2}$ of $\overline{P}_0$ such that $T(e_i) = \lambda_i e_i$. For any collection of real numbers $(\mu_1, \ldots, \mu_{n-2})$, we define a map $\varphi : \overline{P}_0 \approx \mathbb{R}^{n-2} \to \mathcal{A}(v_0)$ by $\varphi(\sum t_i e_i) = g$ such that $u(g) = \sum t_i e_i$, $v(g) = \hat{T}(u(g)) = \sum \lambda_i t_i e_i$ and $\mu(g) = \sum t_i \mu_i + \sum \lambda_i (t_i(1-t_i)/2)$. It is easy to check that for any $(\mu_1, \ldots, \mu_{n-2})$, φ is a morphism.

Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be generators of $\Gamma \approx \mathbf{Z}^k$. We can select $(\mu_1, \ldots, \mu_{n-2})$ so that for every j, if $u(\gamma_j) = \sum t_i e_i$, then $v(\gamma_j) = T(u(\gamma_j)) = \sum \lambda_i t_i e_i$ and $\mu(\gamma_j) = \sum t_i \mu_i + \sum \lambda_i (t_i(1-t_i)/2)$. Then, for every element γ of Γ, we have $\varphi(u(\gamma)) = \gamma$: the image of φ is a Lie abelian subgroup A of $\mathcal{A}(v_0)$ containing Γ and isomorphic to $\overline{P}_0 \approx \mathbb{R}^{n-2}$ considered as a group.

Observe that there is a vector w in $\overline{P}_0$ such that for every i, $\langle u_i|w\rangle = -\mu_i - \lambda_i/2$. Therefore, according to Remark 9.2, after a coordinate change, we can actually assume $\mu_i = -\lambda_i/2$. In this coordinate system, the μ-component of $\varphi(\sum t_i e_i)$ is $-\sum(\lambda_i/2)t_i^2$.

In summary, we have proven that Γ is precisely as described in the introduction for the definition of unipotent space–times. Moreover, our choice of $\hat{T}$ in Proposition 9.5 guarantees that $\tilde{S}$ is contained in a connected component Ω of $\Omega(A)$-cf. the introduction for the definition of $\Omega(A)$. But in the proof of Proposition 8.12, we observed that every lightlike geodesic contained in $\tilde{M}$ intersect $\tilde{S}$: hence, $\mathcal{D}(\tilde{M})$ is contained in $\pi_0^{-1}(\pi_0(\tilde{S}))$. It follows that $\mathcal{D}(\tilde{M})$ is contained in Ω, i.e., that M can be embedded in the unipotent space–time $\Gamma \setminus \Omega$. This achieves the proof of Theorem 1.1 in the nonloxodromic parabolic case.

10. The loxodromic case

It is the case where Γ contains a loxodromic element γ_0. We can then define the convex domain $\Omega_{\mathrm{lox}}(\Gamma)$ which is the interior of the intersection between all the $U(\gamma)$ for *loxodromic* elements of Γ.

Lemma 10.1. *If Γ contains loxodromic elements, $\Omega_{\mathrm{lox}}(\Gamma) = \Omega(\Gamma)$.*

Proof. The inclusion $\Omega(\Gamma) \subset \Omega_{\mathrm{lox}}(\Gamma)$ is obvious. The reverse inclusion would fail if for some nonloxodromic element γ_0 we had $\Omega_{\mathrm{lox}}(\Gamma) \cap U(\gamma_0) \neq \Omega_{\mathrm{lox}}(\Gamma)$. Such a γ_0 cannot be tangent parabolic or spacelike elliptic since in these cases $U(\gamma_0)$ is the entire Minkowski space. According to Remark 8.9, this γ_0 is actually linear parabolic. Then, $U(\gamma_0)$ is the complement of some lightlike hyperplane $P(\gamma_0)$. Thus, since we assume $\Omega_{\mathrm{lox}}(\Gamma) \cap U(\gamma_0) \neq \Omega_{\mathrm{lox}}(\Gamma)$, the convex domain $\Omega_{\mathrm{lox}}(\Gamma)$ contains some γ_0-invariant nonempty open subset U of $P(\gamma_0)$. Now, $P(\gamma_0)$ contains an isotropic direction Δ_0 such that for every element x of $P(\gamma_0) \setminus \Delta_0$, the convex hull of the γ_0-orbit of x is a complete lightlike line Δ (with direction Δ_0). If x is selected in $U \setminus \Delta_0$, Δ is contained in the convex domain $U_{\mathrm{lox}}(\Gamma)$. But this is impossible since the achronal domain $U(\gamma)$ of a loxodromic element cannot contain an entire lightlike affine line. □

Lemma 10.2. *$\Omega(\tilde{S})$ is a regular convex domain.*

Proof. The achronal domain of a loxodromic element γ admits two connected component, one future complete, the other, past complete: we denote them respectively by $U^+(\gamma)$, $U^-(\gamma)$. Since $\Omega(\tilde{S})$ is connected, it is contained in only one of these components. Up to time reversing isometries, we can assume that there is some loxodromic element γ_0 for which $\Omega(\tilde{S}) \subset U^+(\gamma_0)$.

Claim: for every loxodromic element γ, $\Omega(\tilde{S})$ is contained in $U^+(\gamma)$.

Indeed, assume the existence of some γ_1 for which $\Omega(\tilde{S}) \subset U^-(\gamma_1)$. Then, $\Omega(\tilde{S})$ must be contained in $U^+(\gamma_0) \cap U^-(\gamma_1)$. But the reader can easily check that in all cases, there is always a timelike line avoiding the intersection $U^+(\gamma_0) \cap U^-(\gamma_1)$. This is a contradiction with Corollary 8.2.

It follows that $\Omega(\tilde{S})$ is the interior of the intersection of all $U^+(\gamma)$ when γ describes all the loxodromic elements of Γ. Denote by Λ_0 the set of repulsive fixed points of loxodromic elements of Γ in $\mathcal{J}$: we have just proved that $\Omega(\tilde{S})$ is the interior of the future complete convex domain defined by Λ_0. Denote now by $\Lambda(\Gamma)$ the closure of Λ_0 in $\mathcal{J}$. Then, according to Lemma 4.6, $\Omega(\Lambda(\Gamma))$ contains $\Omega(\tilde{S})$: in particular, it is nonempty. Then, according to Lemma 4.8, $\Lambda(\Gamma)$ is future regular, i.e., $\Omega(\Lambda(\Gamma))$ is a future complete regular convex domain. The lemma follows then from Lemma 4.9. □

Corollary 10.3. *The action of Γ on $\Omega(\tilde{S})$ is free and properly discontinuous, and the quotient space $M(S) = \Gamma \setminus \Omega(\tilde{S})$ is a Cauchy-complete globally hyperbolic space–time.*

Proof. According to Propositions 10.2 and 4.16, the action of Γ on $\Omega(\tilde{S})$ is properly discontinuous. By definition of $\Omega(\tilde{S})$, this action is free. The corollary thus follows from Proposition 4.16. □

We may naively think that we achieved the proof of the main theorem, but some point is missing: $M(S)$ is not a Cauchy-hyperbolic space–time if the linear part Γ is not discrete in $\mathrm{SO}_0(1, n-1)$.

10.1. The elementary case

Here, we consider the case where, up to finite index, $L(\Gamma)$ preserves some isotropic direction Δ_0. Since γ_0 is loxodromic, it admits a second isotropic fixed direction: we denote this fixed direction by Δ_1. As in the nonloxodromic case, there are induced actions of Γ on

Q_0, Q_1 the quotient spaces of $\mathbf{M}^n$ by Δ and $\Delta_0^\perp$. The difference with the nonloxodromic case is that the induced action on Q_0 is now given by:

$$\alpha(\gamma)(z, y) = (R(\gamma)(z) + u(\gamma) + yv(\gamma), \mathrm{e}^{a(\gamma)}y + b(y)).$$

The isometry γ is loxodromic if and only if $a(\gamma) \neq 0$.

We choose as origin of Q_1 the fixed point of γ_0. Then, $b(\gamma_0) = 0$, and $U(\gamma_0)$ does not contain the lightlike hyperplane $\pi_1^{-1}(0)$. Since $I = \pi_1(\tilde{S})$, $I' = \pi_1(\Omega(\tilde{S}))$ are $\rho(\gamma_0)$-invariant intervals of $\mathbb{R}$, they are both the interval $]-\infty, 0[$ or $]0, +\infty[$; let say, $I = I' =]0, +\infty[$. Moreover, $b(\gamma) = 0$ for every γ in Γ.

Let Γ_0 be the kernel of a: this is a discrete group of isometries of $\mathbf{M}^n$. For any element γ' of Γ_0, and any element γ of Γ, the conjugate $\gamma\gamma'\gamma^{-1}$ is given by:

$$\alpha(\gamma\gamma'\gamma^{-1})(z, y) = (R(\gamma)R(\gamma')R(\gamma)^{-1}(z) + R(\gamma)(u(\gamma')) + \mathrm{e}^{-a(\gamma)}v(\gamma'), y).$$

Since $\alpha(\Gamma')$ is discrete, it follows that every $v(\gamma') = 0$ since we can choose as γ a loxodromic element for which $a(\gamma) \neq 0$. In other words, elements of Γ_0 are all elliptic.

The nondiscrete part Γ_0^{nd} of Γ_0 has finite index in Γ_0 and is preserved by conjugacy under Γ: the unique maximal Γ_0^{nd}-unipotent subspace $\mathbf{U}$ is preserved by Γ.

Now, we observe that Γ_0 preserves $\tilde{S}$, has no loxodromic elements, and acts trivially on Q_1: we recover the purely unipotent case. In particular, $\mathbf{U}$ is timelike. Since it is Γ-invariant, up to a linear twisted product, we can assume thanks to Lemma 5.5 that $\mathbf{U}$ is the entire Minkowski space and that elements of Γ_0^{nd} are spacelike translations.

Consider now the action of $L(\Gamma)$ on the hyperbolic space $\mathbf{H}^{n-1}$: since $L(\Gamma_0^{nd})$ is trivial, $L(\Gamma_0)$ is a finite group. Therefore, $\mathcal{H}$, the set of $L(\Gamma_0)$-fixed points, is a $L(\Gamma)$-invariant totally geodesic subspace of $\mathbf{H}^{n-1}$. It is not reduced to a point since the isotropic direction Δ_0 defines a $L(\Gamma_0)$-fixed point in $\partial\mathbf{H}^{n-1}$. Now, $L(\Gamma_0)$ acts trivially on $\mathcal{H}$, and for any element γ of Γ, the commutator $\gamma\gamma_0\gamma^{-1}\gamma_0^{-1}$ is in the kernel of a, i.e., in Γ_0. Therefore, $L(\gamma)$ is an isometry of the hyperbolic space $\mathcal{H}$ commuting with the loxodromic element $L(\gamma_0)$: hence, it preserves the *two* fixed points of $L(\gamma_0)$ in $\partial\mathcal{H}$. In other words, there is another isotropic direction Δ_1 preserved by the entire group $L(\Gamma)$.

Let P be the spacelike direction $\Delta_0^\perp \cap \Delta_1^\perp$, and E the quotient space of $\mathbf{M}^n$ by P. Then, Γ acts naturally on the 2-plane E, preserving the Lorentzian metric induced from the metric on $\mathbf{M}^n$. Denote by $\pi_P : \mathbf{M}^n \to E$ the quotient map. We parameterize P by coordinates x, y so that the induced Lorentzian metric is $\mathrm{d}x\,\mathrm{d}y$, and that the fixed γ_0-invariant lightlike hyperplanes are $\pi_P^{-1}(x = 0)$, $\pi_P^{-1}(y = 0)$. Then, since the linear parts of loxodromic elements of Γ preserve the same isotropic directions than $L(\gamma_0)$, the achronal domain of any loxodromic element γ of Γ is $\pi_P^{-1}(\{(x - x(\gamma))(y - y(\gamma)) < 0\})$ for some real numbers $x(\gamma)$, $y(\gamma)$. According to Lemma 10.1, it follows that $\Omega(\Gamma) = \Omega_{\mathrm{lox}}(\gamma)$ has also the expression $\pi_P^{-1}(\{(x - x_0)(y - y_0) < 0\})$. Finally, since $\Omega(\Gamma)$ is γ_0-invariant, we have $x_0 = y_0 = x(\gamma) = y(\gamma) = 0$.

Then $\pi_P^{-1}(0)$ is a Γ-invariant spacelike subspace on which Γ acts by isometries. In other words, there is a coordinate system (x, z, y) on $\mathbf{M}^n$ such that the Lorentzian quadratic form is $xy + |z|^2$ (where $||$ is an Euclidean norm) and such that elements of Γ acts according to the law:

$$\gamma(x, z, y) = (e^{-a(\gamma)}x, R(\gamma)(z) + u(\gamma), e^{a(\gamma)}y).$$

As in Corollary 8.4, we see that the action on $\pi_P^{-1}(0)$ is free and properly discontinuous. Once more, an appropriate use of Bieberbach's theorem leads to the conclusion: the quotient space $M(S) = \Gamma \setminus \Omega(\tilde{S})$ is a linear twisted product over a Misner space–time.

10.2. The non-elementary case

The last case to consider is the case where no finite index subgroup of $L(\Gamma)$ fixes an isotropic direction.

Apply Theorems 5.1, 5.2: the non-discrete part Γ_{nd} is nilpotent; and there is a unique maximal $\mathbf{U}$ on which the action of Γ_{nd} is unipotent. Being unique, $\mathbf{U}$ is also preserved by Γ.

Proposition 10.4. $\mathbf{U}$ *is timelike.*

Proof. If it is lightlike, then it contains a unique isotropic direction which is Γ-invariant: contradiction.

If $\mathbf{U}$ is spacelike, then, according to Corollary 8.4, a finite index subgroup of Γ is abelian. Now, an abelian subgroup of $SO_0(1, n-1)$ fixes a pair of isotropic directions, except if it consists only one elliptic elements fixing one and only one timelike vector v_0. The first case is excluded by hypothesis, and the second case is forbidden too, since in this case v_0 has to be fixed by the entire group Γ, and, thus, to belong to $\mathbf{U}$, which is a contradiction with the definition of $\mathbf{U}$. □

We denote by $L_U(\gamma)$ the linear part of the restriction of γ to $\mathbf{U}$.

Proposition 10.5. $L_U(\Gamma)$ *is discrete.*

Proof. Since the action of Γ on $\mathbf{M}^n/\mathbf{U}$ is a linear Euclidean action, the closure of $L_U(\Gamma)$ in $SO_0(\mathbf{U})$ is the restriction to $\mathbf{U}$ of the closure of $L(\Gamma)$ in $SO_0(1, n-1)$. Hence, the neutral component of this closure is contained in $L_U(\Gamma_{nd})$, proving that this closure is nilpotent and its elements does not have elliptic parts: it contains only hyperbolic or unipotent elements. In particular, if it is not trivial, its center contains a parabolic or hyperbolic element. If it contains a parabolic element γ_0, the unique isotropic direction of γ_0 is $L_U(\Gamma)$-invariant, and thus $L(\Gamma)$-invariant. Contradiction.

If the center contains a hyperbolic element γ_0, then the two isotropic fixed points of $L(\gamma_0)$ has to be exchanged by every element of $L(\Gamma)$. Contradiction. □

In other words, $L_U(\Gamma)$ is a non-elementary Kleinian group. Actually, thanks to Proposition 5.5, we can restrict our study to the case $\mathbf{U} = \mathbf{M}^n$ (be aware that $\Omega_{lox}(\Gamma)$ is obviously equal to $\mathbf{U}^\perp \oplus \Omega_{lox}(\Gamma_{|\mathbf{U}})$, where $\Gamma_{|\mathbf{U}}$ denotes the restriction of Γ to $\mathbf{U}$).

During the proof of Lemma 10.2, we actually proved that $\Omega(\tilde{S})$ is the regular convex domain defined by the closure of repulsive points of loxodromic elements of Γ. Therefore, it coincides precisely with the regular convex domain $\Omega^+(\Lambda(\rho))$ appearing in Definition 4.18.

Hence, it seems that we have all elements in hand to conclude, but one of them is missing: L might be noninjective! Let $\mathcal{N}$ be its kernel: its elements are translations, and the translation vectors form a lattice in some spacelike linear space E. The action of Γ on $\mathcal{N}$ by conjugacy is linear; more precisely, the conjugacy by γ maps the translation by vector v to the translation by vector $L(\gamma)v$. Hence, it extends to an isometric action on the Euclidean space E, which moreover preserves the lattice $\mathcal{N}$. Hence, replacing Γ by some finite index subgroup, we can assume that this action is trivial. Hence, $0 \to \mathcal{N} \to \Gamma \to L(\Gamma) \to 0$ is a central extension.

Consider $p : \mathbf{M}^n \to Q$, the quotient of $\mathbf{M}^n$ by the spacelike subspace E:Q is naturally equipped with a Minkowski metric, and there is a p-equivariant action of Γ on Q which reduces to a (nonlinear) action of $L(\Gamma) \approx \Gamma/\mathcal{N}$. $L(\Gamma)$ is still a discrete group of isometries of Q, but now with injective linear part morphism.

On the other hand, since it is convex, and since it is preserved by translations in $\mathcal{N}$, the regular convex domain $\Omega(\tilde{S})$ is preserved by translations by vectors in E. Therefore, denoting $\Omega(L(\Gamma)) = p(\Omega(\tilde{S}))$, we have $\Omega(\tilde{S}) = p^{-1}(\Omega(L(\Gamma)))$. Finally, since $\Omega(\tilde{S})$ is the regular convex domain defined in $\mathbf{M}^n$ by the closure of repulsive fixed points of $L(\Gamma)$, it should be clear to the reader that $\Omega(L(\Gamma))$ is actually *in the Minkowski space E* the regular convex domain $\Omega(\Lambda(L(\Gamma)))$ as defined for the definition of Cauchy-hyperbolic space–times (Definition 4.18). Hence, the quotient $M(L) = L(\Gamma) \setminus \Omega(L(\Gamma))$ is a Cauchy-hyperbolic space–time.

The quotient map p induces now an isometric fibration $\overline{p} : M(S) \to M(L)$, with fibers isometric to the torus $E/\mathcal{N}$.

Conclusion: $M(S) = \Gamma \setminus \Omega(\tilde{S})$ is a twisted product of a Cauchy-hyperbolic space–time by a flat torus.

11. Summary of the proof

The proof of Theorem 1.1 is quite intricate, and maybe not so easy to follow. Thus, we consider useful to summarize here these proofs, and to add some comments.

11.1. Proof of Theorem 1.1

We start with a Cauchy-complete GH space–time M, with complete Cauchy-surface S. In Section 8, we proved that the developing map $\mathcal{D} : \tilde{M} \to \mathbf{M}^n$ is injective, identifying $\tilde{M}$ with an open domain contained in a open convex domain $\Omega(\tilde{S})$, this last domain being $\rho(\Gamma)$-invariant, where $\rho : \Gamma \to \mathrm{Isom}(\mathbf{M})^n$ is the holonomy morphism. Moreover, it is proved that ρ is injective: Γ is then identified with its image $\rho(\Gamma)$. These preliminaries would conclude if $M(S) = \Gamma \setminus \Omega(\tilde{S})$ was a model space–time, but this is not true in general.

In Section 9, we consider the case where Γ has no loxodromic elements. We prove the folkloric fact that the linear part $L(\Gamma)$ is then elementary, i.e., preserves (up to finite index) a point in $\overline{\mathbb{H}}^{n-1}$ (this fact is maybe most known when the group is discrete). Then, we prove that, if M is not a linear twisted product over a translation space–time, then it is a linear twisted product over some space–time M' for which the holonomy group is an abelian group acting unipotently. All the difficulty is to extend this holonomy group to an abelian Lie group A of unipotent elements acting suitably on $\Omega(\tilde{S})$ (Proposition 9.5). It is then straightforward to prove that $\Omega(\tilde{S})$ is contained in a connected component Ω of $\Omega(A)$. Therefore, M' embeds isometrically in the unipotent space–time $\Gamma \setminus \Omega$.

The nonloxodromic case being ruled out, we assume then the existence of a loxodromic element. We prove that the convex domain $\Omega(\tilde{S})$ is a regular convex domain, so that the quotient $M(S)$ is indeed a space–time, containing an embedded copy of M. We consider once more a dichotomy:

- either $L(\Gamma)$ (up to index 2) preserves a point in $\partial\mathbb{H}^{n-1}$: $M(S)$ is then a linear twisted product over a Misner space–time (Section 10.1).
- either $L(\Gamma)$ is nonelementary: then, $M(S)$ is a linear twisted product over a space–time satisfying the same properties, with the additional requirement that the linear part of the holonomy group is discrete. We then prove that it admits a fibration by flat tori over a Cauchy-hyperbolic space–time (Section 10.2).

11.2. Proof of Theorem 1.2

Reconsider the proof above when M is assumed Cauchy-compact. A fundamental observation is that in a GH space–time, every closed spacelike hypersurface is necessarily a Cauchy surface, hence, every GH space–time containing an embedded copy of M is necessarily Cauchy-compact.

It follows that nontrivial linear twisted products destroy Cauchy-compactness, thus the "up to linear twisted products" appearing in Theorem 1.1 can be erased in the Cauchy-compact version 1.2. As observed previously, unipotent space–times cannot be Cauchy-compact; thus, together with their finite coverings, they disappear in the Cauchy-compact version.

Remark 11.1. In Remark 8.13, we claimed that for maximal Cauchy-compact space–times, $\Omega(\tilde{S})$ is the Cauchy domain of $\tilde{S}$. This should be obvious to the reader for the case of translation space–times, and in the case of Misner space–times and twisted products over standard space–times, it follows from [8] and Lemma 10.2.

11.3. Absolute maximality

Actually, proofs of Theorems 1.1 and 1.2 are not complete. Indeed, they also include the following statements:

(i) Translation space–times, Misner space–times, and twisted products of Cauchy-hyperbolic space–times by flat tori are absolutely maximal, and any unipotent space–time can be tamely embedded in some absolutely maximal unipotent space–time.
(ii) Any maximal Cauchy-compact GH space–time is not only embedded in a finite quotient of a translation space–time, a Misner space–time or the twisted product of a standard space–time by a flat torus, it is actually *isometric* to one of them.

Actually, (ii) follows from (i), since, as noted previously, closed spacelike hypersurfaces in GH space–times are always Cauchy hypersurfaces. In other words, maximal Cauchy-compact GH space–times are also absolutely maximal.

Let us prove (i). To avoid repetitions, we call *model space–times* the space–times listed in (i). Let M_0 be one of them, and consider an isometric embedding $f : M_0 \to M$ in some GH space–time M. Then, according to 1.1, some finite covering M' of M tamely embeds in some model space–time M_1. Hence, some finite covering M'_0 of M_0 embeds isometrically in M_1. By construction, every M_i is the quotient of some open domain convex Ω_i of $\mathbf{M}^n$ by the holonomy group Γ_i. The embedding of M'_0 in M_1 lifts to an isometric embedding $F : \Omega_0 \to \Omega_1$. As any locally defined isometry between open subsets of $\mathbf{M}^n$, F extends to a bijective isometry of the entire Minkowski space: we thus can assume that it is the identity map. Therefore, $\Omega_0 \subset \Omega_1$. Moreover, the fundamental group $\Gamma'_0 \subset \Gamma_0$ of M'_0 is then identified with a subgroup of Γ_1.

Now, remember that F (=id) is the lifting of the *embedding* of M'_0 into M_1. Hence, if the inclusion $\Omega_0 \subset \Omega_1$ is actually surjective, then $\Gamma'_0 = \Gamma_1$, and f is a surjective embedding.

Thus, our remaining task is to establish $\Omega_0 = \Omega_1$, except for the unipotent space–times.

If M_0 is a translations space–time: then, Ω_0 is the entire Minkowski space. The equality $\Omega_0 = \Omega_1$ follows.

If M_0 is loxodromic (*Misner or twisted product of a Cauchy-hyperbolic space–time by a flat torus*): then, Γ_1 contains also loxodromic elements. By construction, Ω_i is the regular convex domain associated to repulsive fixed points of loxodromic elements of Γ_i. Hence, $\Omega_1 \subset \Omega_0$ follows from the inclusion $\Gamma'_0 \subset \Gamma_1$.

Let us now restrict our discussion on the remaining case, i.e. the case where M_0 is unipotent. Then, since Ω_1 contains Ω_0 which is a domain in $\mathbf{M}^n$ between two degenerate hyperplanes, or the half-space defined by a degenerate hyperplane, it is obvious that M_1 is either a translation space–time, or an unipotent space–time. But the first case is excluded since Γ_1 contains parabolic elements (the parabolic elements of Γ_0). Hence, M_1 is actually an unipotent space–time.

In Section 9.2.3, we defined the following objects associated to unipotent space–times:

- the spacelike linear space E_i generated by the u-components of elements of Γ_i,
- a linear map $T_i : E_i \to \mathbb{R}^{n-2}$ expressing the v-components of elements of Γ_i from their u-components.

Moreover, Ω_i has the form $\pi_1^{-1}(I_i)$ where π_1 is the projection map along Γ_i-invariant degenerate hyperplanes from $\mathbf{M}^n$ onto a one-dimensional linear space Q_i.

Consider first the case where I_0 is a bounded interval $]y_-, y_+[$. As previously, we assume without loss of generality the equality $y_+ = -y_-$. In Proposition 9.3, we proved that the operator norm of T_0 is less than or equal to $1/y_+$. Denote by Y_+ the inverse of this operator norm. Then, Proposition 9.5 implies that T extends to some symmetric operator $\hat{T}$, which provides some abelian Lie group A containing Γ_0 and for which $\Omega = \pi_1^{-1}(]-Y_+, Y_+[)$ is a connected component of $\Omega(A)$.

Then, since $y_+ \leq Y_+$, M_0 tamely embeds in the quotient $\Gamma_0 \setminus \Omega$. On the other hand, we claim that this last quotient is absolutely maximal. Indeed, reconsider all the reasoning above, but now assuming $M_0 = \Gamma_0 \setminus \Omega$, i.e., $y_+ = Y_+$, or, equivalently, $\Omega_0 = \Omega$. Consider $\pi_1(\Omega_1) =]y'_-, y'_+[$. Since Γ_1 contains Γ_0, Proposition 9.3 implies $-Y_+ \leq y'_-$ and $y'_+ \leq Y_+$, i.e., the equality $\Omega_0 = \Omega_1$ establishing the absolute maximality of M_0.

The case where I_0 is not bounded can be treated in a similar way.

12. CMC foliations

Let M be a Cauchy-compact space–time. A *CMC foliation* on M is a codimension 1 foliation of M for which all leaves are spacelike and compact, with constant mean curvature. A *CMC time function* is a submersion $t : M \to \mathbb{R}$ such that:

- t is increasing in time (i.e., its restriction to any future oriented timelike curve is increasing);
- every fiber $t^{-1}(s)$ is a hypersurface with constant mean curvature, the CMC value being s.

Observe that in any space–time admitting a CMC time function, the fibers of this time function are the only CMC hypersurfaces (see e.g. [3]). In particular, the CMC time function, if it exists, is unique, and defines moreover the unique CMC foliation on the space–time.

Theorem 12.1. *Every* (*flat*) *maximal Cauchy-compact GH space–time admits a unique CMC foliation. It admits a CMC time function if and only if it is finitely covered by a Misner space–time, or the twisted product of a standard space–time by a flat torus*; *the CMC time function then takes value in* $]-\infty, 0[$ (*future complete case*) *or* $]0, +\infty[$ (*past complete case*). *Cauchy-compact translations space–times do not admit CMC time functions*, *and any CMC spacelike closed hypersurface is a leaf of the unique CMC foliation*.

Proof. The case of standard space–times is treated in [2]. It is also obvious that twisted products by flat tori preserves the existence of CMC time functions.

In the two elementary cases (translation space–times and Misner space–times) there is an abelian Lie group A of dimension $n-1$ acting by isometries, freely, properly discontinuously, with closed spacelike orbits: these orbits are the leaves of a CMC foliation. In the case of Misner space–times, the orbits of A on the universal covering are product of spacelike hyperbolae in $\mathbf{M}^2$ with Euclidean spaces: there are thus obviously the fibers of some CMC time function.

In the case of translation space–times, the CMC value of every leaf is 0 (they are actually totally geodesic). Let I be the quotient space of the action of A. For any closed spacelike CMC hypersurface S, the projection of S into $I \approx \mathbb{R}$ is a compact interval. It means that S is tangent to at least two A-orbits O^+, O^-, so that S is contained in the future of O^- and in the past of O^+. The max principle for CMC hypersurfaces then implies that the constant mean value of S is bigger than the CMC value of O^+ and less than the CMC value of O^-. Since $O^\pm$ are both totally geodesic, the CMC value for S is actually 0, i.e. S is maximal. Then, S lifts as a maximal spacelike hypersurface $\tilde{S}$ in $\mathbf{M}^n$: according to [11], $\tilde{S}$ is a parallel spacelike hyperplane. In other words, S is an orbit of A. □

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